ANALYTICITY AND UNIFORM STABILITY OF THE INVERSE SINGULAR STURM–LIOUVILLE SPECTRAL PROBLEM

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Abstract. We prove that the potential of a Sturm–Liouville operator depends analytically and Lipschitz continuously on the spectral data (two spectra or one spectrum and the corresponding norming constants). We treat the class of operators with real-valued distributional potentials in $W_2^{s-1}(0,1)$, $s \in [0,1]$.

1. Introduction

In this paper we shall establish analyticity and uniform stability of solutions of two inverse spectral problems for a certain class of Sturm–Liouville operators on the interval $[0,1]$. The (direct) spectral problems to be considered are

$$-y''(x) + q(x)y(x) = \lambda y(x), \tag{1.1}$$

subject to some boundary conditions, where $\lambda \in \mathbb{C}$ is the spectral parameter and q is a real-valued potential that might be regular (i.e. integrable) or singular (e.g. a distribution). For simplicity, we shall restrict ourselves to the cases of the Dirichlet boundary conditions

$$y(0) = y(1) = 0 \tag{1.2}$$

and the Dirichlet–Neumann boundary conditions

$$y(0) = y'(1) = 0, \tag{1.3}$$

although other boundary conditions can be treated similarly. (We note here that the derivative in (1.3) must be replaced with a quasi-derivative if q is singular, see Section 2 for details.) We shall always denote by $\lambda_1 < \lambda_2 < \dots$ the eigenvalues of problem (1.1), (1.2) and by $\mu_1 < \mu_2 < \dots$ those of problem (1.1), (1.3).

In 1946, Borg [7] proved that the spectrum of problem (1.1) corresponding to one fixed set of boundary conditions, e.g., the Dirichlet (1.2) or the Dirichlet–Neumann (1.3) ones, does not determine the potential q uniquely. (An exceptional situation where only one spectrum—namely, that for the Neumann boundary conditions—determines the problem was pointed out in 1929 by Ambartsumyan [3].) However, two such spectra already suffice to reconstruct the potential q unambiguously, as follows from the inverse spectral theory for Sturm–Liouville and Schrödinger operators that emerged in the early 1950-ies from the work of Gelfand and Levitan [14], Marchenko [37], and Krein [33] and has been extensively developed in many directions since then. This theory gives an efficient algorithm reconstructing the potential q from the spectra $(\lambda_n)_{n\in\mathbb{N}}$ and $(\mu_n)_{n\in\mathbb{N}}$ and also describes completely the set of possible spectra. For instance, a typical result [38, Thm. 3.4.1] reads

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Theorem A. *For real numbers $\lambda_1 < \lambda_2 < \dots$ and $\mu_1 < \mu_2 < \dots$ to give all the Dirichlet and Dirichlet–Neumann eigenvalues of the Sturm–Liouville problem* (1.1) *with a real-valued $q \in L_2(0,1)$, it is necessary and sufficient that these numbers*

(i) *interlace, i.e., $\mu_n < \lambda_n < \lambda_{n+1}$ for all $n \in \mathbb{N}$, and*
(ii) *have the representation*

$$\lambda_n = \pi^2 n^2 + A + a_n, \qquad \mu_n = \pi^2 (n - \tfrac12)^2 + A + b_n,$$

where $A \in \mathbb{R}$ and $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ are some ℓ_2-sequences.

The induced mapping from the spectral data $\bigl((\lambda_n),(\mu_n)\bigr)$ into the potentials q provides a solution to the inverse spectral problem and has been extensively studied in the literature. In particular, this mapping is shown to be locally continuous in a certain sense, which yields local stability of the inverse spectral problem, see, e.g., [2, 7, 12, 17, 20–23, 36, 40, 43, 44, 46, 48–50, 60] and the references therein. Here we introduce a metric on the set of the spectral data $\bigl((\lambda_n),(\mu_n)\bigr)$ by e.g. identifying such data with the triplets $\bigl(A,(a_n),(b_n)\bigr) \in \mathbb{R} \times \ell_2 \times \ell_2$ in the representation of item (ii) above. Typically, this local stability states that, for a fixed $M > 0$, there are positive ε and L with the following property: if potentials q_1 and q_2 are such that $\|q_1\|_{L_p(0,1)} \le M$ and $\|q_2\|_{L_p(0,1)} \le M$ and the corresponding spectral data $\boldsymbol{\nu}_1 := \bigl((\lambda_{1,n}),(\mu_{1,n})\bigr)$ and $\boldsymbol{\nu}_2 := \bigl((\lambda_{2,n}),(\mu_{2,n})\bigr)$ satisfy $\|\boldsymbol{\nu}_1 - \boldsymbol{\nu}_2\| \le \varepsilon$, then

$$\|q_1 - q_2\|_{L_p(0,1)} \le L \|\boldsymbol{\nu}_1 - \boldsymbol{\nu}_2\| \tag{1.4}$$

for a suitable $p \in [1,\infty]$. For instance, local stability results with $p = 2$ were established in [43,50] in the regular case $q \in L_2(0,1)$, and in [4,8,43] for impedance Sturm–Liouville operators. The cases $p \ge 2$ and $p = \infty$ were treated in [23] and in [17, 44] respectively; earlier Hochstadt in [21, 22] proved stability if only finitely many eigenvalues in one spectrum are changed. The papers [20, 48] studied to what extent only finitely many eigenvalues in one or both spectra determine the potential, and the latter problem in the non-self-adjoint setting was recently discussed in [40]. In [36], stability of reconstruction of general first-order systems from various given data was investigated. Also, stability of the inverse spectral problems on semi-axis was proved in [39, 49], and the inverse scattering and inverse resonance problems on the line and half-line were studied in [10, 19] and [32, 40, 42] respectively.

However, the above results cannot be considered satisfactory, as they refer to the norm of the potential q to be recovered and thus specify neither the allowed noise level ε nor the Lipschitz constant L. Therefore we need a global stability result that asserts (1.4) whenever the spectral data $\boldsymbol{\nu}_1$ and $\boldsymbol{\nu}_2$ run through bounded sets $\mathscr{N}$ and with L only depending on $\mathscr{N}$.

Such global stability of the inverse problem of reconstructing the potential q of a Sturm–Liouville equation from its Dirichlet and Dirichlet–Neumann spectra was recently established by Savchuk and Shkalikov in the paper [57]. In fact, the authors considered therein the class of problems with distributional q in the Sobolev space $W_2^{s-1}(0,1)$, $s \in (0,1]$. Two typical and most important examples of singular potentials belonging to $W_2^{s-1}(0,1)$ only if $s < \frac12$ are the Dirac delta-function $\delta(\cdot - a)$ and the Coulomb-type interaction $1/(\cdot - a)$, $a \in (0,1)$. Recently, singular Sturm–Liouville operators of various types including e.g. operators with distributional potentials in $W_2^{s-1}(0,1)$ [25, 28, 29, 51–55] and operators in impedance form [1, 4, 8, 47] have attracted considerable interest, which is partly motivated by their importance for many applied problems in classical and quantum mechanics, scattering theory etc. Both the

direct and the inverse spectral theory of such singular operators were developed in detail; we refer the reader to e.g. [25, 53, 54, 56].

The purpose of the present paper is two-fold. Firstly, we give an alternative proof of the global stability of the inverse spectral problem for Sturm–Liouville operators with distributional potentials in $W_2^{s-1}(0,1)$ for $s \in [0,1]$, i.e., including the extreme case $s = 0$ (Theorem 2.1). In fact, we prove that the inverse spectral mapping is analytic and locally Lipschitz continuous. Secondly, we prove analogous results for the inverse spectral problem of reconstructing the potential q from the corresponding Dirichlet spectrum (λ_n) and norming constants (α_n) defined in Section 2 (Theorem 2.2); the local stability in this setting was established e.g. by McLaughlin in [43]. Our approach is completely different from that of the paper [43] (which follows the method developed by Pöschel and Trubowitz in the book [46]) and of the paper [57] (which uses the modified Prüfer angle) and is based on a generalization of the classical Gelfand–Levitan–Marchenko method as developed e.g. in [25]. The proof essentially relies on the fact that certain sets of sines and cosines form Riesz bases of $L_2(0,1)$ with uniformly bounded upper and lower bounds [24]. As in [57] we prove first the required results for the extreme cases $s = 0$ and $s = 1$ and then use the Tartar nonlinear interpolation [58] to cover the intermediate s.

The paper is organised as follows. In the next section we formulate the main results and define suitable topologies on the sets of spectral data. Section 3 describes the reconstruction method based on the Gelfand–Levitan–Marchenko integral equation. Theorem 2.2 on analyticity and local Lipschitz continuity of the inverse spectral mapping using the Dirichlet spectrum and norming constants is proved in Section 4, and Section 5 studies dependence of the norming constants on two spectra. Finally, Appendices A, B, and C contain some auxiliary results on Riesz bases, Sobolev spaces, and Fourier transforms therein.

Notations. Throughout the paper, $\sqrt{z}$ shall denote the principal branch of the square root that takes positive values for $z > 0$. For a Hilbert space H, we denote by $\mathscr{B}(H)$ the algebra of bounded linear operators acting in H.

2. Preliminaries and main results

In this section we define explicitly the class of Sturm–Liouville operators to be studied and state the main results for the inverse problems.

2.1. **The operators.** Given a real-valued distribution q in $W_2^{-1}(0,1)$, we define Sturm–Liouville operators T corresponding to the differential expression

$$-\frac{d^2}{dx^2} + q \tag{2.1}$$

by means of regularisation by quasi-derivatives suggested by Shkalikov and Savchuk [52, 53]. We take a real-valued distributional primitive $\sigma \in L_2(0,1)$ of q and set

$$l_\sigma(y) := -(y' - \sigma y)' - \sigma y'$$

for $y \in W_2^1(0,1)$ such that the *quasi-derivative* $y^{[1]} := y' - \sigma y$ is absolutely continuous and $l_\sigma(y)$ is in $L_2(0,1)$. We then define the operators $T_{\mathrm{D}} = T_{\mathrm{D}}(\sigma)$ and $T_{\mathrm{N}} = T_{\mathrm{N}}(\sigma)$ as the restrictions of l_σ onto the functions satisfying the boundary conditions $y(0) = y(1) = 0$ and $y(0) = y^{[1]}(1) = 0$ respectively.

Since $l_\sigma(y) = -y'' + qy$ in the sense of distributions, these operators coincide with the classical Sturm–Liouville operators if $q \in L_1(0,1)$. Moreover, the operators $T_{\mathrm{D}}(\sigma)$ and $T_{\mathrm{N}}(\sigma)$ depend continuously on $\sigma \in L_2(0,1)$ in the uniform resolvent sense [52, 53].

Therefore $T_{\mathrm{D}}(\sigma)$ and $T_{\mathrm{N}}(\sigma)$ are the most natural Sturm–Liouville operators associated with differential expression (2.1) for $q \in W_2^{-1}(0,1)$.

We observe that although the differential expression l_σ is independent of the particular choice of the primitive σ of q, the boundary conditions for the operator $T_{\mathrm{N}}(\sigma)$ and the norming constants for $T_{\mathrm{D}}(\sigma)$ introduced below do depend on σ rather than on q. Therefore it is the function σ, and not q, that has to be reconstructed in the inverse spectral problem for singular Sturm–Liouville operators under consideration. We shall call σ the *regularized potential* of the Sturm–Liouville operators $T_{\mathrm{D}}(\sigma)$ and $T_{\mathrm{N}}(\sigma)$.

2.2. **Spectral data.** It is known [52, 53] that for a real-valued $\sigma \in L_2(0,1)$ the operators $T_{\mathrm{D}}(\sigma)$ and $T_{\mathrm{N}}(\sigma)$ are self-adjoint, bounded below, and have simple discrete spectra. We denote by $\lambda_1 < \lambda_2 < \dots$ the eigenvalues of the operator T_{D} and by $\mu_1 < \mu_2 < \dots$ those of T_{N} and recall that these eigenvalues interlace, i.e., $\mu_n < \lambda_n < \mu_{n+1}$ for all $n \in \mathbb{N}$, and satisfy the relations

$$\sqrt{\lambda_n} = \pi n + \rho_{2n}, \qquad \sqrt{\mu_n} = \pi(n - \tfrac{1}{2}) + \rho_{2n-1} \tag{2.2}$$

with some ℓ_2-sequence (ρ_n) [52, 53]. If q belongs to $W_2^{s-1}(0,1)$ for some $s \in [0,1]$, then by [29] there exists a unique function $\sigma^* \in W_2^s(0,1)$ such that $\rho_n = s_n(\sigma^*)$, where

$$s_n(f) := \int_0^1 f(x) \sin \pi n x \, dx$$

is the n-th sine Fourier coefficient of f.

In the paper [54], the mapping $\mathscr{F}_{\sin} : f \mapsto (s_n(f))_{n=1}^\infty$ defined on the Sobolev spaces $W_2^s(0,1)$ was studied in detail for all $s \ge 0$. For $s \in [0,1]$ the results of [54] can be specified as follows. Denote by $W_{2,0}^1(0,1)$ the subspace of $W_2^1(0,1)$ consisting of functions that vanish at the endpoints $x = 0$ and $x = 1$ and set

$$W_{2,0}^s(0,1) := \bigl[W_{2,0}^1(0,1), L_2(0,1)\bigr]_s, \qquad s \in (0,1),$$

to be the interpolation space [35, Ch. I.9]. In particular, for $s < \frac{1}{2}$ the space $W_{2,0}^s(0,1)$ coincides with $W_2^s(0,1)$, for $s > \frac{1}{2}$ it is the proper subspace of the latter consisting of functions vanishing at the endpoints, and $W_{2,0}^{1/2}(0,1)$ is a proper subspace of $W_2^{1/2}(0,1)$ defined by more complicated conditions. We also set

$$\ell_2^s := \{\mathbf{x} = (x_n)_{n=1}^\infty \mid \|\mathbf{x}\|_s^2 := \sum n^{2s} |x_n|^2 < \infty\};$$

this is a Hilbert space under the scalar product $(\mathbf{x}, \mathbf{y})_s := \sum n^{2s} x_n \overline{y_n}$ for $\mathbf{x} := (x_n)_{n=1}^\infty$ and $\mathbf{y} := (y_n)_{n=1}^\infty$. The interpolation theory then shows that for all $s \in [0,1]$ the mapping $\mathscr{F}_{\sin}$ is an isomorphism between $W_{2,0}^s(0,1)$ and ℓ_2^s; moreover, under an equivalent norm on $W_{2,0}^s(0,1)$ this mapping becomes isometric.

We set $P_0(x) := 1 - x$ and $P_1(x) = x$; then for an arbitrary $f \in W_2^1(0,1)$ the function

$$f_0(x) := f(x) - f(0)P_0(x) - f(1)P_1(x)$$

belongs to $W_{2,0}^1(0,1)$; thus $\mathscr{F}_{\sin}(f) \in \ell_2^1 \dotplus \operatorname{ls}\{\mathbf{e}_0, \mathbf{e}_1\}$, with

$$\mathbf{e}_j := \mathscr{F}_{\sin}(P_j) = \bigl((-1)^{j(n-1)}/(\pi n)\bigr)_{n=1}^\infty$$

and $\operatorname{ls} S$ standing for the linear span of a set S. We now set $\hat{\ell}_2^s := \ell_2^s$ for $s \in [0, \frac{1}{2})$ and

$$\hat{\ell}_2^s := \ell_2^s \dotplus \operatorname{ls}\{\mathbf{e}_0, \mathbf{e}_1\}$$

for $s \in [\frac{1}{2}, 1]$; in this latter case the scalar product in $\hat{\ell}_2^s$ is introduced via

$$\bigl(\mathbf{x} + a_0\mathbf{e}_0 + a_1\mathbf{e}_1, \mathbf{y} + b_0\mathbf{e}_0 + b_1\mathbf{e}_1\bigr)_s := (\mathbf{x}, \mathbf{y})_s + a_0\overline{b_0} + a_1\overline{b_1}.$$

Then by Lemma 1 of [54] we see that $\mathscr{F}_{\sin}$ extends to an isomorphism of $W_2^s(0,1)$ and $\hat\ell_2^s$ for all $s \in [0,1]$, which, moreover, is even isometric under an equivalent norm on $W_2^s(0,1)$. Therefore, for $q \in W_2^{s-1}(0,1)$ the sequence (ρ_n) defined by (2.2) belongs to $\hat\ell_2^s$.

Next we define the norming constants. For a nonzero $z \in \mathbb{C}$, we denote by $y(\cdot,z)$ a solution of the equation $l_\sigma(y) = z^2 y$ satisfying the initial conditions $y(0) = 0$ and $y^{[1]}(0) = z$. Then $y(\cdot,\sqrt{\lambda_n})$ is an eigenfunction corresponding to the eigenvalue λ_n of the operator T_{D}, and we call the number $\alpha_n := \|y(\cdot,\sqrt{\lambda_n})\|_{L_2}^{-2}$ the *norming constant* for this eigenvalue. It is known [28] that if $q \in W_2^{s-1}(0,1)$ for $s \in [0,1]$, then

$$\alpha_n = 2 + \beta_{2n},$$

where β_{2n} is the $2n$-th cosine Fourier coefficient $c_{2n}(\tilde\sigma)$ for a unique function $\tilde\sigma \in W_2^s(0,1)$ of zero mean that is even with respect to $x = \frac12$, i.e., $\tilde\sigma(1-x) = \tilde\sigma(x)$; here we set

$$c_n(f) := \int_0^1 f(x)\cos \pi n x\, dx.$$

By the arguments similar to the above the mapping $\mathscr{F}_{\cos} : f \mapsto \bigl(c_n(f)\bigr)_{n=1}^\infty$ is an isomorphism between the subspace $\widetilde{W}_{2,\mathrm{even}}^s(0,1)$ of even (with respect to $x = \frac12$) functions in $W_2^s(0,1)$ of zero mean and the subspace $\ell_{2,\mathrm{even}}^s$ of sequences in ℓ_2^s with vanishing odd entries. Clearly, the spaces $\ell_{2,\mathrm{even}}^s$ and ℓ_2^s are isomorphic.

The norming constants α_n can be determined from the spectra of the operators T_{D} and T_{N} as follows. We set $S(z) := y(1,z)$ and $C(z) := y^{[1]}(1,z)$. Due to the integral representations (3.1) and (3.2) below, S and C are entire functions of exponential type 1 with zeros $0, \pm\sqrt{\lambda_n}$ and $0, \pm\sqrt{\mu_n}$ respectively. The Hadamard canonical products of S and C are

$$S(z) = z\prod_{n=1}^\infty \frac{\lambda_n - z^2}{\pi^2 n^2}, \qquad C(z) = z\prod_{n=1}^\infty \frac{\mu_n - z^2}{\pi^2 (n-\frac12)^2}, \tag{2.3}$$

so that S and C are uniquely determined by their zeros. Finally, we have [26]

$$\alpha_n = \frac{2\sqrt{\lambda_n}}{\dot S(\sqrt{\lambda_n})C(\sqrt{\lambda_n})}, \tag{2.4}$$

where the dot denotes the derivative in z.

2.3. **Main results.** Without loss of generality, we may consider only uniformly positive operators, adding the number $\mu_1 + 1$ to q and all the Dirichlet and Dirichlet–Neumann eigenvalues as required. Respectively, we introduce the set $\mathscr{N}^s$ of data $\bigl((\lambda_n)_{n\in\mathbb{N}}, (\mu_n)_{n\in\mathbb{N}}\bigr)$ with the following properties:

- (N1) $\mu_1 \ge 1$ and the sequences (λ_n) and (μ_n) strictly interlace, i.e., $\mu_n < \lambda_n < \mu_{n+1}$ for all $n \in \mathbb{N}$;
- (N2) the numbers $\rho_{2n} := \sqrt{\lambda_n} - \pi n$ and $\rho_{2n-1} := \sqrt{\mu_n} - \pi(n-\frac12)$, $n \in \mathbb{N}$, form a sequence $(\rho_n)_{n\in\mathbb{N}}$ in $\hat\ell_2^s$.

In this way every element $\boldsymbol{\nu} := (\boldsymbol{\lambda}, \boldsymbol{\mu})$ of $\mathscr{N}^s$ is identified with a sequence (ρ_n) in $\hat\ell_2^s$ or, equivalently, with a unique function $f \in W_2^s(0,1)$ satisfying $s_n(f) = \rho_n$. This induces a topology on $\mathscr{N}^s$; moreover, $\|(\rho_n)\|_s$ or $\|f\|_{W_2^s}$ define equivalent metrics on $\mathscr{N}^s$.

According to [28], every element of $\mathscr{N}^s$ consists of eigenvalue sequences of the operators $T_{\mathrm{D}}(\sigma)$ and $T_{\mathrm{N}}(\sigma)$ corresponding to some real-valued regularized potential $\sigma \in$

$W_2^s(0,1)$ and, conversely, for every real-valued $\sigma \in W_2^s(0,1)$ with $T_\mathrm{N}(\sigma) \ge I$ the corresponding spectral data form an element of $\mathscr{N}^s$. When the regularized potential σ varies over a bounded subset of $W_2^s(0,1)$, then the main theorem of the paper [55] implies that the corresponding spectral data $\big((\lambda_n),(\mu_n)\big)$ remain in a bounded subset of $\mathscr{N}^s$. Moreover, the Prüfer angle technique used in [55] yields then a positive h such that all the corresponding spectral data $\big((\lambda_n),(\mu_n)\big)$ are *h-separated*, i.e., such that the inequalities $\sqrt{\mu_{n+1}}-\sqrt{\lambda_n} \ge h$ and $\sqrt{\lambda_n}-\sqrt{\mu_n} \ge h$ hold for every $n \in \mathbb{N}$. Summarizing, we conclude that the uniform stability of the inverse spectral problem we would like to establish is only possible on the convex closed sets $\mathscr{N}^s(h,r)$ of the spectral data consisting of all elements of $\mathscr{N}^s$ that are h-separated and satisfy $\|(\rho_n)\|_s \le r$.

In these notations, one of the main results of the paper reads as follows.

Theorem 2.1. *For every $s \in [0,1]$, $h \in (0,\pi/2)$, and $r > 0$, the mapping*

$$\mathscr{N}^s(h,r) \ni \boldsymbol{\nu} \mapsto \sigma \in W_2^s(0,1)$$

is analytic and Lipschitz continuous.

Lipschitz continuity means that there exists a number $L = L(s,h,r)$ such that for any two elements $\boldsymbol{\nu}_1$ and $\boldsymbol{\nu}_2$ of $\mathscr{N}^s(h,r)$ the regularized potentials σ_1 and σ_2 in $W_2^s(0,1)$ solving the inverse spectral problems for the data $\boldsymbol{\nu}_1$ and $\boldsymbol{\nu}_2$ satisfy

$$\|\sigma_1 - \sigma_2\|_{W_2^s(0,1)} \le L\|\boldsymbol{\nu}_1 - \boldsymbol{\nu}_2\|_{\mathscr{N}^s}.$$

See [9] for definitions and properties of analytic mappings between Banach spaces.

In fact, we prove first analyticity and local Lipschitz continuity in the inverse spectral problem of reconstructing σ from the spectrum (λ_n) of $T_\mathrm{D}(\sigma)$ and the norming constants (α_n) (see Theorem 2.2 below), and then derive Theorem 2.1 by showing that the norming constants depend analytically and locally Lipschitz continuously on the two spectra.

More exactly, we denote by $\mathscr{L}^s$ the family of strictly increasing sequences $\boldsymbol{\lambda} := (\lambda_n)$ for which the sequence (ρ_n) with $\rho_{2n-1} = 0$ and $\rho_{2n} := \sqrt{\lambda_n} - \pi n$ forms an element of $\hat\ell_2^s$ and introduce the topology on $\mathscr{L}^s$ by identifying such $\boldsymbol{\lambda}$ with $(\rho_n) \in \hat\ell_2^s$. It follows from the results of Subsection 2.2 that the subspace $\hat\ell^s_{2,\mathrm{even}}$ of $\hat\ell_2^s$ consisting of elements with vanishing odd entries and the subspace $W^s_{2,\mathrm{odd}}(0,1)$ of functions in $W_2^s(0,1)$ which are odd with respect to $x = \frac12$ are isomorphic under the sine Fourier transform $\mathscr{F}_{\sin}$. For $h \in (0,\pi)$ and $r > 0$, we denote by $\mathscr{L}^s(h,r)$ the closed convex subset of $\mathscr{L}^s$ consisting of sequences (λ_n) with $\lambda_1 \ge 1$, $\sqrt{\lambda_{n+1}} - \sqrt{\lambda_n} \ge h$, and such that $\|(\rho_n)\|_s \le r$.

Next, we write $\mathscr{A}^s$ for the set of sequences $(\alpha_n)_{n\in\mathbb{N}}$ of positive numbers for which the sequence $(\beta_n)_{n\in\mathbb{N}}$, with $\beta_{2n-1} = 0$ and $\beta_{2n} := \alpha_n - 2$, belongs to $\ell^s_{2,\mathrm{even}}$. This induces the topology of ℓ_2^s on $\mathscr{A}^s$; by the results of Subsection 2.2 the space $\ell^s_{2,\mathrm{even}}$ (and thus $\mathscr{A}^s$) is homeomorphic to the subspace $\widetilde{W}^s_{2,\mathrm{even}}(0,1)$ of functions in $W_2^s(0,1)$ of zero mean which are even with respect to $x = \frac12$. We further consider closed subsets $\mathscr{A}^s(h,r)$ of $\mathscr{A}^s$ consisting of all (α_n) satisfying the inequality $\alpha_n \ge h$ for all $n \in \mathbb{N}$ and such that $\|(\beta_n)\|_s \le r$.

It follows from [28] that to every $(\boldsymbol{\lambda},\boldsymbol{\alpha}) \in \mathscr{L}^s \times \mathscr{A}^s$ there corresponds a unique regularized real-valued potential $\sigma \in W_2^s(0,1)$ such that $\boldsymbol{\lambda}$ and $\boldsymbol{\alpha}$ are the sequences of eigenvalues and the corresponding norming constants of the operator $T_\mathrm{D}(\sigma)$. The more elaborate properties of this mapping are given by the following theorem.

Theorem 2.2. *For every $s \in [0,1]$, $h \in (0,\pi)$, $h' \in (0,2)$, and any positive r and r', the inverse spectral mapping*

$$\mathscr{L}^s(h,r) \times \mathscr{A}^s(h',r') \ni \big(\boldsymbol{\lambda}, \boldsymbol{\alpha}\big) \mapsto \sigma \in W_2^s(0,1)$$

is analytic and Lipschitz continuous.

We conclude this section by observing that analyticity and other properties of the direct spectral mapping

$$W_2^s(0,1) \ni \sigma \mapsto (\boldsymbol{\lambda}, \boldsymbol{\mu}, \boldsymbol{\alpha}) \in \mathscr{N}^s \times \mathscr{A}^s,$$

at least for the classical case $s=1$, are well known and studied in detail in, e.g., [46].

3. Solution of the inverse spectral problem via the GLM equation

In this section, we recall shortly the method of reconstructing the regularized potential σ based on the Gelfand–Levitan–Marchenko (GLM) equation [25]; see also [54] for an alternative approach. Using this method, we shall later show analyticity and Lipschitz continuity of the inverse spectral mapping.

We recall that $y(\cdot,z)$ stands for the solution of the equation $l_\sigma(y) = z^2 y$ satisfying the initial conditions $y(0,z)=0$ and $y^{[1]}(0,z) = z$. This function has the representation [27]

$$y(x,z) = \sin zx + \int_0^x k(x,t) \sin z(1-2t)\, dt, \tag{3.1}$$

where the kernel k is lower-triangular (i.e., $k(x,t)=0$ a.e. in the domain $0 \le x \le t \le 1$) and has the property that, for every fixed $x \in [0,1]$, the functions $k(\cdot,x)$ and $k(x,\cdot)$ are in $L_2(0,1)$ and depend continuously therein on $x \in [0,1]$. Also,

$$y^{[1]}(x,z) = z\cos zx + z\int_0^x k_1(x,t)\cos z(1-2t)\, dt \tag{3.2}$$

where a kernel k_1 has similar properties. Denoting by K the integral operator with kernel k and by I the identity operator in $L_2(0,1)$, we see that $I+K$ is the transformation operator mapping solutions of the unperturbed ($\sigma=0$) differential equation $l_0(y)=z^2y$ to those of $l_\sigma(y) = z^2 y$.

The GLM equation relates the spectral data for the operator $T_{\mathrm{D}} = T_{\mathrm{D}}(\sigma)$ (i.e., its eigenvalues and norming constants) with the transformation operator $I+K$. To derive it, we start with the resolution of identity for T_{D},

$$I = \operatorname*{s-lim}_{N\to\infty} \sum_{n=1}^N \alpha_n(\,\cdot\,, y_n) y_n,$$

where s-lim stands for the limit in the strong operator topology, $(\cdot,\cdot)$ is the scalar product in $L_2(0,1)$, and $y_n := y(\cdot, \sqrt{\lambda_n})$. Recalling that $y_n = (I+K)s_n$ with $s_n(x) = \sin\sqrt{\lambda_n}x$, we get

$$I = (I+K)\Big[\operatorname*{s-lim}_{N\to\infty}\sum_{n=1}^N \alpha_n(\,\cdot\,,s_n)s_n\Big](I+K^*), \tag{3.3}$$

The operator in the square brackets has the form $I+F$, where $F = F(\boldsymbol{\lambda},\boldsymbol{\alpha})$ is an integral operator of Hilbert–Schmidt class $\mathfrak{S}_2$ with kernel

$$f(x,t) := \tfrac12\big(\phi(x+t) - \phi(|x-t|)\big), \tag{3.4}$$

where

$$\phi(x) := \sum_{n\in\mathbb{N}} \bigl(2\cos\pi n x - \alpha_k \cos\sqrt{\lambda_n}x\bigr) \tag{3.5}$$

is a function in $L_2(0,2)$. Applying $(I+K^*)^{-1}$ to both sides of (3.3) and rewriting the resulting relation in terms of the kernels k and f, we get the GLM equation

$$k(x,t) + f(x,t) + \int_0^x k(x,\xi) f(\xi,t)\, d\xi = 0, \qquad x \ge t. \tag{3.6}$$

The algorithm of reconstructing q from the spectral data $\bigl((\lambda_n),(\mu_n)\bigr)$ proceeds now as follows. We first calculate the numbers α_n via (2.4), then construct the function ϕ of (3.5), form the kernel f of (3.4), solve the GLM equation (3.6) for k, and set

$$\sigma(x) := -\phi(2x) - 2\int_0^x k(x,\xi) f(\xi,x)\, d\xi. \tag{3.7}$$

Then $\sigma \in L_2(0,1)$ is the unique regularized potential for which the operators $T_{\mathrm{D}}(\sigma)$ and $T_{\mathrm{N}}(\sigma)$ have eigenvalues λ_n and μ_n, $n\in\mathbb{N}$, respectively, see [25,26]. Moreover, if k and ϕ are smooth, then the GLM equation implies that $\sigma(x) = 2k(x,x) - \phi(0)$, thus yielding the classical relation

$$q(x) = 2\frac{d}{dx}k(x,x)$$

for the potential q. It was proved in [28] that if the sequences (λ_n) and (μ_n) are such that the corresponding sequence (ρ_n) is in $\hat\ell_2^s$, with $s\in(0,1]$, then σ of (3.7) belongs to $W_2^s(0,1)$, i.e., the reconstructed potential q is in $W_2^{s-1}(0,1)$.

4. Reconstruction from a spectrum and norming constants

In this section, we prove Theorem 2.2 on analyticity and Lipschitz continuity in the inverse spectral problem of reconstructing the regularized potentials of Sturm–Liouville differential expressions from their Dirichlet spectra and the corresponding norming constants.

We shall study the correspondence between the data $(\boldsymbol{\lambda},\boldsymbol{\alpha}) \in \mathscr{L}^s(h,r)\times\mathscr{A}^s(h',r')$ and the regularized potentials $\sigma\in W_2^s(0,1)$ of the Sturm–Liouville operator $T_{\mathrm{D}}(\sigma)$ through the chain of mappings

$$(\boldsymbol{\lambda},\boldsymbol{\alpha}) \mapsto \phi \mapsto F \mapsto K \mapsto \sigma,$$

in which $\phi\in L_2(0,2)$ is the function of (3.5), F is the operator with kernel f of (3.4), $K\in\mathfrak{S}_2$ is the integral operator with kernel k that solves the GLM equation (3.6), and, finally, σ is given by (3.7). Throughout this section, we fix $h\in(0,\pi)$, $h'\in(0,2)$, and positive r and r', and shall always denote by λ_n the elements of a sequence $\boldsymbol{\lambda}\in\mathscr{L}^s(h,r)$ and by α_n those of an $\boldsymbol{\alpha}\in\mathscr{A}^s(h',r')$.

Lemma 4.1. *The mapping*

$$\mathscr{L}^s(h,r)\times\mathscr{A}^s(h',r') \ni (\boldsymbol{\lambda},\boldsymbol{\alpha}) \mapsto \phi \in W_2^s(0,2)$$

is analytic and Lipschitz continuous.

Proof. We have $\phi(2x) = \varphi_{\boldsymbol{\lambda}}(x) + \psi_{\boldsymbol{\lambda},\boldsymbol{\alpha}}(x)$, where

$$\varphi_{\boldsymbol{\lambda}}(x) := 2\sum_{n=1}^\infty [\cos(2\pi n x) - \cos(2\sqrt{\lambda_n}x)]$$

and

$$\psi_{\boldsymbol{\lambda},\boldsymbol{\alpha}}(x) := -\sum_{n=1}^{\infty} \beta_n \cos(2\sqrt{\lambda_n}x),$$

with $\beta_n := \alpha_n - 2$. We recall that $\sqrt{\lambda_n} = \pi n + s_{2n}(f)$ for a unique $f \in W^s_{2,\mathrm{odd}}(0,1)$ and $\beta_n = c_{2n}(g)$ for a unique $g \in \widetilde{W}^s_{2,\mathrm{even}}(0,1)$ and that the mapping $(\boldsymbol{\lambda},\boldsymbol{\alpha}) \mapsto (f,g)$ is isomorphic from $\mathscr{L}^s \times \mathscr{A}^s$ into $W^s_{2,\mathrm{odd}}(0,1) \times \widetilde{W}^s_{2,\mathrm{even}}(0,1)$. Therefore analyticity and Lipschitz continuity of the mapping under consideration follows from Corollary C.3. $\square$

It is advantageous to regard the GLM equation (3.6) as the relation between the integral operators K and F generated by the kernels k and f. We shall need several related notions, which we now recall.

The ideal $\mathfrak{S}_2$ of Hilbert–Schmidt operators consists of integral operators with square summable kernels, and the scalar product $\langle X, Y\rangle_2 := \mathrm{tr}(XY^*)$ introduces a Hilbert space structure on $\mathfrak{S}_2$. For an integral operator $T \in \mathfrak{S}_2$ with kernel t we find that $\langle T, T\rangle_2 = \int_0^1\int_0^1 |t(x,y)|^2\,dx\,dy$; thus the estimate

$$\begin{aligned}\int_0^1 |f(x,y)|^2\,dx &\le \tfrac12\int_0^1 |\phi(x+y)|^2\,dx + \tfrac12\int_0^1 |\phi(|x-y|)|^2\,dx\\ &\le \int_0^2 |\phi(\xi)|^2\,d\xi\end{aligned} \tag{4.1}$$

shows that $F \in \mathfrak{S}_2$ and $\|F\|^2_{\mathfrak{S}_2} = \langle F, F\rangle_2 \le \|\phi\|^2_{L_2(0,2)}$. Moreover, the mapping $\phi \mapsto F$ is linear (and thus analytic and Lipschitz continuous) from $L_2(0,2)$ into $\mathfrak{S}_2$.

Further, we denote by $\mathfrak{S}_2^+$ the subspace of $\mathfrak{S}_2$ consisting of all Hilbert–Schmidt operators with lower-triangular kernels. In other words, $T \in \mathfrak{S}_2$ belongs to $\mathfrak{S}_2^+$ if the kernel t of T satisfies $t(x,y) = 0$ a.e. outside the domain $\Omega^+ := \{(x,y) \mid 0 < y < x < 1\}$. For an arbitrary $T \in \mathfrak{S}_2$ with kernel t the cut-off t^+ of t given by

$$t^+(x,y) = \begin{cases} t(x,y) & \text{for } x \ge y,\\ 0 & \text{for } x < y,\end{cases}$$

generates an operator $T^+ \in \mathfrak{S}_2^+$, and the corresponding mapping $\mathcal{P}^+ : T \mapsto T^+$ turns out to be an orthoprojector in $\mathfrak{S}_2$ onto $\mathfrak{S}_2^+$, i.e. $(\mathcal{P}^+)^2 = \mathcal{P}^+$ and $\langle \mathcal{P}^+X, Y\rangle_2 = \langle X, \mathcal{P}^+Y\rangle_2$ for any $X, Y \in \mathfrak{S}_2$; see details in [16, Ch. I.10].

With these notations, the GLM equation (3.6) can be recast as

$$K + \mathcal{P}^+F + \mathcal{P}^+(KF) = 0 \tag{4.2}$$

or

$$(\mathcal{I} + \mathcal{P}^+_F)K = -\mathcal{P}^+F,$$

where $\mathcal{P}^+_X$ is the linear operator in $\mathfrak{S}_2$ defined by $\mathcal{P}^+_X Y = \mathcal{P}^+(YX)$ and $\mathcal{I}$ is the identity operator in $\mathfrak{S}_2$. Therefore solvability of the GLM equation is strongly connected with the properties of the operator $\mathcal{P}^+_F$.

Lemma 4.2. *For every $X \in \mathscr{B}(L_2(0,1))$, the operator $\mathcal{P}^+_X$ is bounded in $\mathfrak{S}_2$. Moreover, for every F from the set*

$$\mathcal{F} := \{F(\boldsymbol{\lambda},\boldsymbol{\alpha}) \mid (\boldsymbol{\lambda},\boldsymbol{\alpha}) \in \mathscr{L}^s(h,r) \times \mathscr{A}^s(h',r')\} \tag{4.3}$$

the operator $\mathcal{I} + \mathcal{P}^+_F$ is invertible in $\mathscr{B}(\mathfrak{S}_2^+)$ and the inverse $(\mathcal{I} + \mathcal{P}^+_F)^{-1}$ depends analytically and Lipschitz continuously in $\mathscr{B}(\mathfrak{S}_2^+)$ on $F \in \mathcal{F}$ in the topology of $\mathfrak{S}_2$.

Proof. Boundedness of $\mathcal{P}_X^+$ in $\mathfrak{S}_2$ is a straightforward consequence of the inequality
$$\|\mathcal{P}_X^+ Y\|_{\mathfrak{S}_2} \le \|YX\|_{\mathfrak{S}_2} \le \|X\|_{\mathscr{B}(L_2(0,1))}\|Y\|_{\mathfrak{S}_2},$$
see [15, Ch. 3]. Assume next that $I + X \ge \varepsilon I$ in $L_2(0,1)$; then, for any $Y \in \mathfrak{S}_2^+$,
$$\langle(\mathcal{I}+\mathcal{P}_X^+)Y, Y\rangle_2 = \langle Y, Y\rangle_2 + \langle YX, Y\rangle_2 = \operatorname{tr}\bigl(Y(I+X)Y^*\bigr).$$
We see that $Y(I+X)Y^* \ge \varepsilon YY^*$ and by monotonicity of the trace we get
$$\langle(\mathcal{I}+\mathcal{P}_X^+)Y, Y\rangle_2 \ge \varepsilon\langle Y, Y\rangle_2,$$
i.e., $\mathcal{I}+\mathcal{P}_X^+ \ge \varepsilon\mathcal{I}$ in $\mathfrak{S}_2^+$.

Now, if $F = F(\boldsymbol{\lambda}, \boldsymbol{\alpha})$ is constructed as explained in Section 3 from $\boldsymbol{\lambda} = (\lambda_k)_{k\in\mathbb{N}} \in \mathscr{L}^s(h,r)$ and $\boldsymbol{\alpha} = (\alpha_k)_{k\in\mathbb{N}} \in \mathscr{A}^s(h',r')$, then
$$I + F = \operatorname*{s-lim}_{N\to\infty} \sum_{k=1}^{N} \alpha_k(\,\cdot\,, s_k)s_k$$
with $s_k(x) := \sin\sqrt{\lambda_k}x$. By definition, $\alpha_k \ge h'$ for all $k \in \mathbb{N}$; moreover, by Theorem A.1 the sequence $(s_k)_{k\in\mathbb{N}}$ forms a Riesz basis of the space $L_2(0,1)$ and its lower bound $m > 0$ can be chosen the same for all $\boldsymbol{\lambda} \in \mathscr{L}^s(h,r)$. Therefore
$$\bigl((I+F)y, y\bigr) = \sum_{k=1}^{\infty} \alpha_k|(y, s_k)|^2 \ge h'm\|y\|^2$$
for every $y \in L_2(0,1)$, so that $I + F \ge h'mI$.

By the above, $\mathcal{I}+\mathcal{P}_F^+ \ge h'm\mathcal{I}$ in $\mathfrak{S}_2^+$; thus $\mathcal{I}+\mathcal{P}_F^+$ is boundedly invertible in $\mathscr{B}(\mathfrak{S}_2^+)$ and
$$\|(\mathcal{I}+\mathcal{P}_F^+)^{-1}\| \le (h'm)^{-1}.$$
Since $\mathcal{P}_X^+$ depends linearly on X, it follows that the mapping
$$F \mapsto (\mathcal{I}+\mathcal{P}_F^+)^{-1}$$
from $\mathfrak{S}_2$ into $\mathscr{B}(\mathfrak{S}_2^+)$ is analytic and Lipschitz continuous on the set $\mathcal{F}$. The proof is complete. □

Corollary 4.3. *For every $F \in \mathcal{F}$, the GLM equation* (4.2) *has a unique solution*
$$K := -(\mathcal{I}+\mathcal{P}_F^+)^{-1}\mathcal{P}^+F \in \mathfrak{S}_2^+;$$
moreover, the mapping $F \mapsto K$ from $\mathcal{F} \subset \mathfrak{S}_2$ to $\mathfrak{S}_2^+$ is analytic and Lipschitz continuous.

In view of the above results and formula (3.7), the regularized potential σ is determined uniquely by the function ϕ of (3.5). To complete the proof of Theorem 2.2, we shall show that the induced mapping $\phi \mapsto \sigma$ is analytic and locally Lipschitz continuous from the space $W_2^s(0,2)$ into $W_2^s(0,1)$ for every $s \in [0,1]$. We shall establish this for $s = 0$ and $s = 1$, and then interpolate to cover all the intermediate values $s \in (0,1)$.

4.1. **The case $s = 0$.** This case is the easiest to treat, although it corresponds to the set of Sturm–Liouville operators with the most singular potentials—namely, with distributional potentials in $W_2^{-1}(0,1)$.

Lemma 4.4. *The function σ of* (3.7) *depends analytically and Lipschitz continuously in $L_2(0,1)$ on the function $\phi \in L_2(0,2)$ of* (3.5) *that is in the range of the mapping of Lemma 4.1 for $s = 0$.*

Proof. By definition, $\sigma(x) = -\phi(2x) - 2\int_0^x k(x,t)f(t,x)\,dt$, where k is the kernel of the solution K of the GLM equation (3.6) and $f(t,x) = \frac{1}{2}[\phi(t+x) - \phi(|x-t|)]$. Thus the integral above depends linearly on k and ϕ; moreover, in view of (4.1) we get

$$\begin{aligned}
&\int_0^1 dx\Big|\int_0^x k(x,t)f(t,x)\,dt\Big|^2\\
&\qquad \le \int_0^1 dx\int_0^x |k(x,t)|^2\,dt\int_0^x |f(t,x)|^2\,dt\\
&\qquad \le \|\phi\|^2_{L_2(0,2)}\int_0^1\int_0^1 |k(x,t)|^2 dxdt = \|\phi\|^2_{L_2(0,2)}\|K\|^2_{\mathfrak{S}_2}.
\end{aligned}$$

Since $K \in \mathfrak{S}_2^+$ depends analytically and Lipschitz continuously on $F \in \mathcal{F}$ and $F \in \mathfrak{S}_2$ depends linearly and continuously on $\phi = \phi(\boldsymbol{\lambda}, \boldsymbol{\alpha})$, the result follows. $\square$

This completes the proof of Theorem 2.2 for the case $s = 0$.

4.2. **The case $s = 1$.** The function ϕ of (3.5) belongs in this case to $W_2^1(0,2)$, and the solution k to the GLM equation (3.6) must also possess some extra smoothness. We recall that functions in $W_2^1(0,1)$ are continuous and that there is $C > 0$ such that

$$\max_{x\in[0,1]} |g(x)| \le C\|g\|_{W_2^1(0,1)}$$

for every $g \in W_2^1(0,1)$. As above, we denote by Ω^+ the set $\{(x,t) \mid 0 < t < x < 1\}$.

Lemma 4.5. *Let $s = 1$ and ϕ and k be defined as above. Then the distributional derivative $\partial_x k$ of k belongs to $L_2(\Omega^+)$ and the induced mapping $(\boldsymbol{\lambda}, \boldsymbol{\alpha}) \mapsto \partial_x k$ is analytic and Lipschitz continuous from $\mathscr{L}^1(h,r) \times \mathscr{A}^1(h',r')$ into $L_2(\Omega^+)$.*

Proof. Let $(\boldsymbol{\lambda}, \boldsymbol{\alpha})$ be the element of $\mathscr{L}^1(h,r) \times \mathscr{A}^1(h',r')$ that generates the function $\phi \in W_2^1(0,2)$. We set

$$\phi_n(x) := \sum_{k=1}^n \bigl(2\cos\pi kx - \alpha_k\cos\sqrt{\lambda_k}x\bigr),$$

which corresponds to taking $\alpha_k = \alpha_{k,0} := 2$ and $\lambda_k = \lambda_{k,0} := \pi^2k^2$ for $k > n$. Choose n_0 so large that $|\sqrt{\lambda_n} - \pi n| < \pi - h$ if $n > n_0$; then for such n the sequences

$$(\lambda_k)_{k=1}^n \cup (\lambda_{k,0})_{k>n}$$

and

$$(\alpha_k)_{k=1}^n \cup (\alpha_{k,0})_{k>n}$$

belong to $\mathscr{L}^1(h,r)$ and $\mathscr{A}^1(h',r')$ respectively and $\phi_n \to \phi$ in $W_2^1(0,1)$ by Lemma 4.1. We form the kernel f_n taking ϕ_n instead of ϕ in (3.4); then the GLM equation with f replaced by f_n has a unique solution $k_n \in L_2(\Omega^+)$. Denoting by F_n and K_n the integral operators with kernels f_n and k_n, we see that $f_n \to f$ in $L_2\bigl((0,1)\times(0,1)\bigr)$ means that $F_n \to F$ in $\mathfrak{S}_2$; hence $K_n \to K$ in $\mathfrak{S}_2^+$ by Corollary 4.3, i.e., $k_n \to k$ in $L_2(\Omega^+)$.

Since the integral operator F_n is of finite rank, the solution k_n can be written in an explicit form and is easily seen to be smooth in the domain Ω^+, cf. [11, Sect. 12] and [16, Ch. IV.3]. We set $l_n := \partial_x k_n$; then l_n satisfies in Ω^+ the equation

$$l_n(x,t) + \int_0^x l_n(x,\xi)f_n(\xi,t)\,d\xi = -\tilde{f}_n(x,t) - k_n(x,x)f_n(x,t),$$

where $\tilde f_n := \partial_x f_n$. The convergence $\phi_n \to \phi$ in $W_2^1(0,2)$ implies that $f_n \to f$ in $C(\Omega^+)$ and $\tilde f_n(x,t) \to \tilde f(x,t) := \frac12[\phi'(x+t) - \phi'(x-t)]$ in $L_2(\Omega^+)$; also,

$$\sigma_n(x) := 2k_n(x,x) - \phi_n(0) = -\phi_n(2x) - 2\int_0^x k_n(x,t) f_n(t,x)\,dt$$

converge in $L_2(0,1)$ to $\sigma(x)$ by Lemma 4.4. It follows that the kernels

$$g_n(x,t) := \tilde f_n(x,t) + k_n(x,x) f_n(x,t)$$

converge in $L_2(\Omega^+)$, as $n \to \infty$, to

$$g(x,t) := \tilde f(x,t) + \tfrac12[\sigma(x) + \phi(0)] f(x,t),$$

with $\tilde f := \partial_x f$. Denoting by L_n, G_n, and G the integral operators in $\mathfrak{S}_2^+$ with kernels l_n, g_n, and g respectively, we conclude by Lemma 4.2 that

$$L_n = -(\mathcal{I} + \mathcal{P}^+_{F_n})^{-1} G_n \to -(\mathcal{I} + \mathcal{P}^+_F)^{-1} G =: L$$

as $n \to \infty$ in the topology of $\mathfrak{S}_2^+$. Therefore l_n converge in $L_2(\Omega^+)$ to the kernel l of the operator L. We conclude that, in the sense of generalized functions, $l = \partial_x k$ and $\partial_x k$ belongs to $L_2(\Omega^+)$ as claimed. It is easily seen that the mapping $(\boldsymbol\lambda, \boldsymbol\alpha) \mapsto G$ from $\mathscr{L}^1(h,r) \times \mathscr{A}^1(h',r')$ to $\mathfrak{S}_2$ is analytic and Lipschitz continuous. We finally apply Lemma 4.2 to conclude that $L \in \mathfrak{S}_2$ depends in the same manner on $F \in \mathcal{F}$ and $G \in \mathfrak{S}_2$; here $\mathcal{F}$ is the set of (4.3) corresponding to $s = 1$. □

To complete the proof of Theorem 2.2 for $s = 1$, it suffices to show that the function $\sigma_1(x) := \int_0^x k(x,t) f(t,x)\,dt$ depends analytically and Lipschitz continuously in $W_2^1(0,1)$ on $(\boldsymbol\lambda, \boldsymbol\alpha) \in \mathscr{L}^1(h,r) \times \mathscr{A}^1(h',r')$. That σ_1 depends in this manner in $L_2(0,1)$ was established in Subsection 4.1. Also,

$$\sigma_1'(x) = \tfrac14[\sigma(x) + \phi(0)][\phi(2x) - \phi(0)] + \int_0^x l(x,t) f(t,x)\,dt + \int_0^x k(x,t)\tilde f(t,x)\,dt,$$

where, as in the proof of Lemma 4.5, $l(x,t) := \partial_x k(x,t)$ and $\tilde f(t,x) := \partial_x f(t,x)$. Clearly, the first summand above belongs to $L_2(0,1)$ and depends therein analytically and Lipschitz continuously on $(\boldsymbol\lambda, \boldsymbol\alpha)$. Also, l and $\tilde f$ depend in the same manner in $L_2(\Omega^+)$ on the spectral data (the former by Lemma 4.5, the latter by linearity and direct estimates (4.1)). Thus both integrals give functions in $L_2(0,1)$ with required dependence on $(\boldsymbol\lambda, \boldsymbol\alpha)$ (see the proof of Lemma 4.4), which establishes Theorem 2.2 for $s = 1$.

4.3. **The case $s \in (0,1)$.** The general case will be covered by the nonlinear interpolation theorem due to Tartar [58], which implies the following result.

Proposition 4.6. *Assume that (X_0, X_1) and (Y_0, Y_1) are pairs of Banach spaces with continuous embeddings $X_1 \hookrightarrow X_0$ and $Y_1 \hookrightarrow Y_0$. Let also $\Phi : X_0 \to Y_0$ be a nonlinear mapping that is Lipschitz continuous on the balls $B_{X_0}(r) := \{x \in X_0 \mid \|x\|_{X_0} \le r\}$ for every $r > 0$. Assume further that $\Phi X_1 \subset Y_1$ and that Φ is Lipschitz continuous on every ball $B_{X_1}(r)$ of X_1 as a mapping from X_1 into Y_1. Construct the interpolation spaces $X_s := [X_1, X_0]_s$ and $Y_s := [Y_1, Y_0]_s$, $s \in (0,1)$, by the complex interpolation method; then Φ acts boundedly from X_s to Y_s for every $s \in (0,1)$ and, moreover, its restriction to the ball $B_{X_s}(r)$ of X_s is Lipschitz continuous for every $r > 0$. In other words, for every $r > 0$ there is $C = C(r,s)$ such that*

$$\|\Phi(x_1) - \Phi(x_2)\|_{Y_s} \le C\|x_1 - x_2\|_{X_s}$$

whenever x_1 *and* x_2 *belong to* $B_{X_s}(r)$.

By definition, the spaces ℓ_2^s, $\hat\ell_2^s$ and $W_2^s(0,1)$, as well as their "even" and "odd" subspaces, form the interpolation space scales. Therefore the above proposition, in view of the results of Subsections 4.1 and 4.2, implies that for every $h \in (0,\pi)$, $h' \in (0,2)$, positive r and r', and $s \in (0,1)$ the mapping $(\boldsymbol\lambda,\boldsymbol\alpha) \mapsto \sigma$ is Lipschitz continuous from $\mathscr{L}^s(h,r)\times\mathscr{A}^s(h',r')$ to $W_2^s(0,1)$. Analyticity for all $s\in[0,1]$ again follows from that for $s=0$ and $s=1$ and interpolation theorem for linear operators [6,35] applied to the Fréchet derivative of this mapping.

Remark 4.7. For $s<\frac12$, the arguments of the paper [28] combined with the results of the previous subsection provide a direct proof of analytic and Lipschitz continuous dependence of $\sigma\in W_2^s(0,1)$ on the spectral data $(\boldsymbol\lambda,\boldsymbol\alpha)\in\mathscr{L}^s(h,r)\times\mathscr{A}^s(h',r')$.

Remark 4.8. It should be clear how one can iterate the considerations of Subsection 4.2 to all natural values of s and then interpolate as in Subsection 4.3 to get all positive s, cf. [38, Sect. 3.4] and [57].

5. Reconstruction from two spectra

In this section, we complete the proof of Theorem 2.1 by establishing uniform continuity of the norming constants on the two spectra. More exactly, given an element $\big((\lambda_n),(\mu_n)\big)\in\mathscr{N}^s(h,r)$, we define numbers α_n by the relation

$$\alpha_n = \frac{2\sqrt{\lambda_n}}{\dot S(\sqrt{\lambda_n})C(\sqrt{\lambda_n})}, \tag{5.1}$$

where the entire functions S and C are given by the infinite products (2.3), and prove the following theorem.

Theorem 5.1. *For every* $s\in[0,1]$, $h\in(0,\pi/2)$ *and* $r>0$ *the mapping*

$$\mathscr{N}^s(h,r)\ni(\boldsymbol\lambda,\boldsymbol\mu)\mapsto\boldsymbol\alpha\in\mathscr{A}^s$$

is analytic and Lipschitz continuous; moreover, there are $h'>0$ *and* $r'>0$ *such that the range of this mapping belongs to* $\mathscr{A}^s(h',r')$.

It suffices to show that there are $h''>0$ and $r''>0$ such that the numbers $\tilde\beta_n$ defined via the relation

$$\frac{2}{\alpha_n} = 1+\tilde\beta_n$$

satisfy the inequality $1+\tilde\beta_n\ge h''$ and form an ℓ_2^s-sequence $\tilde{\boldsymbol\beta}:=(\tilde\beta_n)$ depending analytically and Lipschitz continuously on $(\boldsymbol\lambda,\boldsymbol\mu)\in\mathscr{N}^s(h,r)$ and having norm not greater than r''. Indeed, then $\alpha_n\ge h':=2/(1+r'')$ and $\beta_{2n}:=\alpha_n-2=-2\tilde\beta_n/(1+\tilde\beta_n)$. We observe that, for a bounded sequence (d_n), the mapping $(x_n)\mapsto(d_nx_n)$ is a bounded linear operator in ℓ_2^s of norm $d:=\sup_n|d_n|$. Since $\big|-2/(1+\tilde\beta_n)\big|\le 2/h''$, we conclude that the sequence (β_{2n}) belongs to ℓ_2^s and has the norm at most $2r''/h''$; the sequence (β_n) with $\beta_{2n-1}=0$ belongs then to $\ell^s_{2,\mathrm{even}}$ and is of norm at most $r':=2^{s/2+1}r''/h''$.

To justify analyticity and Lipschitz continuity of $\boldsymbol\alpha$ in $\mathscr{A}^s$, we exploit the fact that ℓ_2^s is a Banach algebra under the point-wise multiplication $(x_n)\cdot(y_n)=(x_ny_n)$. We denote by A the unital extension of ℓ_2^s; elements of A have the form $a\mathbf{1}+\mathbf{x}$, where $a\in\mathbb{C}$, $\mathbf{1}$ is the sequence with all its elements equal to 1, and $\mathbf{x}=(x_n)\in\ell_2^s$, and the norm in A is given by $\|a\mathbf{1}+\mathbf{x}\|_A:=|a|+\|\mathbf{x}\|_s$. An element $a\mathbf{1}+\mathbf{x}$ is invertible in A if and only if $a\ne0$ and $a+x_n\ne0$ for all $n\in\mathbb{N}$; in this case the inverse is equal to

$\frac{1}{a}\mathbf{1} + \mathbf{y}$ with $\mathbf{y} = (y_n)$ and $y_n := -x_n/[a(a+x_n]$. Since the sequence $\big(1/[a(a+x_n)]\big)$ is bounded, the above reasoning shows that indeed $\mathbf{y}$ belongs to ℓ_2^s.

We now see that

$$\tfrac{1}{2}\boldsymbol{\alpha} = (\mathbf{1} + \tilde{\boldsymbol{\beta}})^{-1};$$

since taking an inverse element is an analytic mapping in a unital Banach algebra, $\boldsymbol{\alpha}$ depends analytically on $\tilde{\boldsymbol{\beta}}$. Lipschitz continuity of $\boldsymbol{\alpha}$ follows from the fact that, for the set of $\tilde{\boldsymbol{\beta}}$ considered, $\mathbf{1} + \tilde{\boldsymbol{\beta}}$ have uniformly bounded inverses in A (of norm not greater than $1 + r''/h''$).

To the rest of this section, we shall use the notations $\omega_{2n} := \sqrt{\lambda_n}$ and $\omega_{2n-1} := \sqrt{\mu_n}$ and $\rho_n := \omega_n - \pi n/2$. In view of (5.1) we have

$$1 + \tilde{\beta}_n = \dot{S}(\omega_{2n})\,\frac{C(\omega_{2n})}{\omega_{2n}} =: \big(1 + a_n\big)\big(1 + b_n\big),$$

where we set

$$a_n := (-1)^n \dot{S}(\omega_{2n}) - 1, \qquad b_n := (-1)^n \frac{C(\omega_{2n})}{\omega_{2n}} - 1. \tag{5.2}$$

Thus we need to prove that both $\dot{S}(\omega_{2n})$ and $C(\omega_{2n})/\omega_{2n}$ are uniformly bounded away from zero, that the sequences (a_n) and (b_n) are the sequences of the even cosine Fourier coefficients of some functions h_1 and h_2 from $W^s_{2,\mathrm{even}}(0,1)$ of zero mean, and that the mappings

$$\mathscr{N}^s(h,r) \ni (\boldsymbol{\lambda}, \boldsymbol{\mu}) \mapsto h_1 \in W^s_{2,\mathrm{even}}(0,1) \tag{5.3}$$

and

$$\mathscr{N}^s(h,r) \ni (\boldsymbol{\lambda}, \boldsymbol{\mu}) \mapsto h_2 \in W^s_{2,\mathrm{even}}(0,1) \tag{5.4}$$

are analytic and Lipschitz continuous. We do this in the two subsections that follow.

5.1. **Analyticity and continuity.**

Lemma 5.2. *The mappings of* (5.3) *and* (5.4) *are analytic and Lipschitz continuous.*

Proof. We observe that in view of (3.1) and (3.2) the functions $\dot{S}(z)$ and $C(z)/z$ have the representation

$$\cos z + \int_0^1 g(t) \cos z(1-2t)\,dt,$$

with $g(t) = (1-2t)k(1,t) \in L_2(0,1)$ for $\dot{S}(z)$ and $g(t) = k_1(1,t) \in L_2(0,1)$ for $C(z)/z$. Therefore both a_n and b_n can be written as

$$[(-1)^n \cos\omega_{2n} - 1] + (-1)^n \int_0^1 g(t) \cos\omega_{2n}(1-2t)\,dt =: d_n + e_n,$$

with respective $g \in L_2(0,1)$. Clearly, we may (and shall) take the even part g_{even} of g instead of g in the above integral.

Recalling that $\omega_{2n} = \pi n + s_{2n}(f)$ for a (unique) function $f \in W^s_{2,\mathrm{odd}}(0,1)$ and observing that $s_{2n}(f) = \mathrm{i}\hat{f}(n)$, with $\hat{f}(n)$ being the n-th Fourier coefficient of a function f (see Appendix B), we can write the d_n as

$$d_n = \cos s_{2n}(f) - 1 = \cosh \hat{f}(n) - 1 = \sum_{k=1}^\infty \frac{\hat{f}(n)^{2k}}{(2k)!}.$$

It follows that the d_n is the n-th Fourier coefficient of the function $\tilde{f}$ given by

$$\tilde{f} := \sum_{k=1}^{\infty} \frac{f^{<2k>}}{(2k)!},$$

where $f^{<k>}$ is the k-fold convolution of f with itself. The function $\tilde{f}$ has zero mean and is even with respect to $x = \frac{1}{2}$ and therefore $d_n = c_{2n}(\tilde{f})$. By the results of Appendix B the mapping $f \mapsto \tilde{f}$ is analytic from $W^s_{2,\text{odd}}(0,1)$ to $W^s_{2,\text{even}}(0,1)$ and is Lipschitz continuous on bounded subsets of f.

Recalling Lemma C.4 and the remark with formula (C.2) following it, we see that the e_n give the $2n$-th cosine Fourier coefficient of the function $h := \frac{1}{2}\Psi(\mathrm{i}f, g)$ in $W^s_{2,\text{even}}(0,1)$ of zero mean, which depends analytically and boundedly Lipschitz continuously on f and g.

It remains to apply Lemma 5.3 below to conclude that the even parts of the functions $(1-2t)k(1,t)$ and $k_1(1,t)$, which we have taken as the function g above, depend analytically and Lipschitz continuously in $W^s_2(0,1)$ on the spectral data in $\mathscr{N}^s(h,r)$. This completes the proof of the lemma. $\square$

Lemma 5.3. *For every $s \in [0,1]$, the mappings*

$$\mathscr{N}^s(h,r) \ni (\boldsymbol{\lambda}, \boldsymbol{\mu}) \mapsto k_{\text{odd}}(1,\cdot) \in W^s_2(0,1)$$

and

$$\mathscr{N}^s(h,r) \ni (\boldsymbol{\lambda}, \boldsymbol{\mu}) \mapsto k_{1,\text{even}}(1,\cdot) \in W^s_2(0,1)$$

are analytic and Lipschitz continuous. Here $k_{\text{odd}}(1,\cdot)$ and $k_{1,\text{even}}(1,\cdot)$ are respectively odd and even parts (with respect to $x = \frac{1}{2}$) of the functions $k(1,\cdot)$ and $k_1(1,\cdot)$.

Proof. Since both mappings are treated similarly, we only consider in detail the first one. By definition of the function $y(\cdot, z)$ we have $y(1, \omega_{2n}) = 0$, and thus the numbers $\omega_{2n} = \pi n + s_{2n}(f)$ along with 0 and $-\omega_{2n}$ are zeros of the entire function

$$y(1,z) = \sin z + \int_0^1 k(1,t) \sin z(1-2t)\,dt.$$

This function does not depend on the even part $k_{\text{even}}(1,\cdot)$ of the function $k(1,\cdot)$; in fact, it can be written in the form

$$\sin z + \int_0^1 g(t) \mathrm{e}^{\mathrm{i}z(1-2t)}\,dt \tag{5.5}$$

with $g(t) := -\mathrm{i}k_{\text{odd}}(1,t)$. Since the set of functions $\{\sin \omega_{2n}(1-2t)\}_{n\in\mathbb{N}}$ is complete in $L_{2,\text{odd}}(0,1)$ (cf. the results of Appendix A), one can show that $k_{\text{odd}}(1,\cdot)$ is uniquely determined by the zeros $\omega_{2n} = \pi n + s_{2n}(f) = \pi n + i\hat{f}(n)$. Some important properties of the induced mapping $f \mapsto g$ can be derived from the paper [30].

Indeed, the results of [30] imply that for every $f \in W^s_2(0,1)$ there exists a unique function $g \in W^s_2(0,1)$ such that all zeros (counting multiplicities) of the entire function (5.5) are given by the numbers $\pi n + \hat{f}(n)$, $n \in \mathbb{Z}$. Such pairs of f and g in fact satisfy the relation

$$H(f,g) := s(f) + g + \sum_{k=1}^{\infty} \frac{(M^k g) * f^{<k>}}{k!} = 0; \tag{5.6}$$

here

$$s(f) := \sum_{k=0}^{\infty} \frac{(-1)^k f^{<2k+1>}}{(2k+1)!},$$

$f^{<k>}$ is the k-fold convolution of f with itself, and M is the operator of multiplication by $\mathrm{i}(1-2x)$. The function H is analytic from $W_2^s(0,1) \times W_2^s(0,1)$ into $W_2^s(0,1)$, and its partial derivatives $\partial_f H(f,g)$ and $\partial_g H(f,g)$ are given by

$$\partial_f H(f,g)(h_1) = \Big(c(f) + \sum_{k=1}^{\infty} \frac{(M^k g) * f^{<k-1>}}{(k-1)!}\Big) * h_1, \tag{5.7}$$

$$\partial_g H(f,g)(h_2) = h_2 + \sum_{k=1}^{\infty} \frac{(M^k h_2) * f^{<k>}}{k!} \tag{5.8}$$

with

$$c(f) := \sum_{k=0}^{\infty} \frac{(-1)^k f^{<2k>}}{(2k)!}.$$

We assume now that $f \in W_{2,\mathrm{odd}}^s(0,1)$ is such that the corresponding sequence $\boldsymbol{\lambda} = (\lambda_n)_{n\in\mathbb{N}}$ with $\lambda_n := \omega_{2n}^2 = (\pi n + s_{2n}(f))^2$ belongs to $\mathscr{L}^s(h,r)$. Set $S_{\boldsymbol{\lambda}}$ to be the function of (2.3); then $S_{\boldsymbol{\lambda}}$ can also be represented as (5.5). Direct calculations show that the n-th Fourier coefficient of the function of (5.7) is equal to

$$(-1)^n \hat{h}_1(n)\Big[\cos\omega_{2n} + \int_0^1 \mathrm{i}(1-2t)g(t)\mathrm{e}^{\mathrm{i}\omega_{2n}(1-2t)}\,dt\Big] = (-1)^n \hat{h}_1(n)\dot{S}_{\boldsymbol{\lambda}}(\omega_{2n}).$$

By Lemma 5.4 below there are numbers K_1 and K_2 such that

$$K_1 \le |\dot{S}_{\boldsymbol{\lambda}}(\omega_{2n})| \le K_2$$

for all $\boldsymbol{\lambda} \in \mathscr{L}^s(h,r)$ and all $n \in \mathbb{N}$. The results of Appendix B imply that the operator $\partial_f H(f,g)$ is bounded in every space $W_2^s(0,1)$ and its norm is at most K_2.

Similarly, the n-th Fourier coefficient of the function of (5.8) is equal to

$$(-1)^n \int_0^1 h_2(t)\mathrm{e}^{\mathrm{i}\omega_{2n}(1-2t)}\,dt.$$

By Theorem A.1 there exist positive M and m such that, for all $\boldsymbol{\lambda} \in \mathscr{L}^s(h,r)$, the sequences $(\mathrm{e}^{\mathrm{i}\omega_{2n}(1-2x)})_{n\in\mathbb{Z}}$ form Riesz bases of $L_2(0,1)$ of upper bound M and lower bound m, see Appendix A. Therefore the operator $H_g := \partial_g H(f,g)$,

$$H_g:\ h_2 \mapsto \sum_{n\in\mathbb{Z}} (-1)^n \big(h_2, \mathrm{e}^{\mathrm{i}\omega_{2n}(2x-1)}\big)\,\mathrm{e}^{2\pi n \mathrm{i}x},$$

is bounded and boundedly invertible in $L_2(0,1)$, with $\|H_g\| \le M^{1/2}$ and $\|H_g^{-1}\| \le m^{-1/2}$. If $h_2 \in W_2^1(0,1)$, then we integrate by parts to get

$$c_n := \big(h_2, \mathrm{e}^{\mathrm{i}\omega_{2n}(2x-1)}\big) = \frac{1}{2\mathrm{i}\omega_{2n}}\big[h_2(0)\mathrm{e}^{\mathrm{i}\omega_{2n}} - h_2(1)\mathrm{e}^{-\mathrm{i}\omega_{2n}}\big] + \frac{1}{2\mathrm{i}\omega_{2n}}\big(h_2', \mathrm{e}^{\mathrm{i}\omega_{2n}(2x-1)}\big).$$

It is clear that the sequence $(c_n)_{n\in\mathbb{Z}}$ forms an element of $\tilde{\ell}_2^1(\mathbb{Z})$, see Section B.2. Thus the operator H_g acts boundedly and boundedly invertible in $W_2^1(0,1)$, and it remains to use the interpolation theorem to derive the same properties of H_g in $W_2^s(0,1)$ for all $s \in [0,1]$.

We now use the implicit mapping theorem to conclude that the mapping $f \mapsto g$ is analytic in $W_2^s(0,1)$. Recalling the isomorphism of the space $\mathscr{L}^s$ of sequences $\boldsymbol{\lambda} = (\lambda_n)$

of the Dirichlet eigenvalues of the Sturm–Liouville operators $T(\sigma)$ with $\sigma \in W_2^s(0,1)$ and the subspace $W^s_{2,\mathrm{odd}}(0,1)$ explained in Section 2, we conclude that the mapping

$$\mathscr{L}^s(h,r) \ni \boldsymbol{\lambda} \mapsto k_{\mathrm{odd}}(1,\cdot) \in W^s_{2,\mathrm{odd}}(0,1)$$

is analytic. The uniform bounds on the inverses of the partial derivatives $\partial_f H(f,g)$ and $\partial_g H(f,g)$ established above imply that this mapping is Lipschitz continuous, and the proof for the first mapping is complete.

The second mapping of the lemma is treated analogously using the relations

$$\cos\omega_{2n-1} + \int_0^1 k_1(1,t)\cos\omega_{2n-1}(1-2t)\,dt = 0$$

and the Riesz basis properties of the system $(\cos\omega_{2n-1}t)_{n\in\mathbb{N}}$, cf. Remark A.2. □

5.2. **Uniform positivity of α_n.** Since $\mathscr{N}^s(h,r)\subset\mathscr{N}(h,r)$ if $s\ge 0$, it only suffices to consider the case $s=0$. In view of formula (5.1), uniform positivity of α_n immediately follows from the lemma below.

Lemma 5.4. *For every $h\in(0,\pi/2)$ and $r>0$ we have*

$$\sup_{(\boldsymbol{\lambda},\boldsymbol{\mu})}\sup_{n\in\mathbb{N}} \log|\dot S(\omega_{2n})| < \infty, \qquad \sup_{(\boldsymbol{\lambda},\boldsymbol{\mu})}\sup_{n\in\mathbb{N}} \log\frac{|C(\omega_{2n})|}{\omega_{2n}} < \infty,$$

where S and C are constructed via (2.3) *from sequences $\boldsymbol{\lambda} := (\omega_{2k}^2)$ and $\boldsymbol{\mu} := (\omega_{2k-1}^2)$, and the suprema are taken over $(\boldsymbol{\lambda},\boldsymbol{\mu})\in\mathscr{N}(h,r)$.*

Proof. By (2.3), we have

$$\dot S(\sqrt{\lambda_n}) = -\frac{2\lambda_n}{\pi^2n^2}\prod_{k\in\mathbb{N},\ k\ne n}\frac{\lambda_k-\lambda_n}{\pi^2k^2}.$$

Dividing both sides by

$$\cos\pi n = \frac{d\sin z}{dz}\Big|_{z=\pi n} = -2\prod_{k\in\mathbb{N},\ k\ne n}\frac{\pi^2k^2-\pi^2n^2}{\pi^2k^2},$$

we conclude that[1]

$$|\dot S(\sqrt{\lambda_n})| = \frac{\lambda_n}{\pi^2n^2}\prod_{k\ne n}\frac{\lambda_n-\lambda_k}{\pi^2n^2-\pi^2k^2} = \prod_{k\in\mathbb{Z},\ k\ne n}\frac{\omega_{2k}-\omega_{2n}}{\pi(k-n)},$$

where we set $\omega_{-k} := -\omega_k$ for $k\in\mathbb{N}$ and $\omega_0 := 0$. Set also (recall that $\rho_k := \omega_k - \pi k$)

$$a_{k,n} := \frac{\omega_{2k}-\omega_{2n}}{\pi(k-n)} - 1 = \frac{\rho_{2k}-\rho_{2n}}{\pi(k-n)}$$

if $k\ne n$ and $a_{n,n} := 0$; then $|\dot S(\omega_{2n})| = \prod_{k\in\mathbb{Z}}(1+a_{k,n})$. Since the sequence (ω_n) is h-separated for every $(\boldsymbol{\lambda},\boldsymbol{\mu})\in\mathscr{N}(h,r)$, we have $1+a_{k,n}\ge 2h/\pi$ for all integer k and n. Therefore, with

$$K := \max_{x\ge -1+2h/\pi}\Big|\frac{\log(1+x)-x}{x^2}\Big| < \infty,$$

we get the estimate

$$\Big|\log\prod_{k\in\mathbb{Z}}(1+a_{k,n})\Big| \le \Big|\sum_{k\in\mathbb{Z}}a_{k,n}\Big| + K\sum_{k\in\mathbb{Z}}a_{k,n}^2, \tag{5.9}$$

[1] In what follows, all summations and multiplications over the index set $\mathbb{Z}$ will be taken in the principal value sense and the symbol V.p. will be omitted.

provided the two series converge.

Clearly,

$$\sum_{k\neq n}\frac{1}{k-n}=0,$$

and thus

$$\Bigl|\sum_{k\in\mathbb{Z}}a_{k,n}\Bigr|=\Bigl|\frac{1}{\pi}\sum_{k\neq n}\frac{\rho_{2k}}{k-n}\Bigr|\le\frac{\sqrt{2}r}{\sqrt{3}}$$

by the Cauchy–Bunyakovski–Schwarz inequality (recall that $\sum_{k\in\mathbb{Z}}\rho_{2k}^2\le 2r^2$ by the definition of the set $\mathscr{N}(h,r)$ and $\sum_{k\neq n}(k-n)^{-2}=\pi^2/3$). Next, the inequality

$$a_{k,n}^2\le\frac{2\rho_{2k}^2}{(k-n)^2}+\frac{2\rho_{2n}^2}{(k-n)^2}$$

for $k\neq n$ yields

$$\sum_{k\in\mathbb{Z}}a_{k,n}^2\le 4r^2\sum_{k\neq n}\frac{1}{(n-k)^2}=\frac{4\pi^2r^2}{3}.$$

It follows from (5.9) that

$$\Bigl|\log\prod_{k\in\mathbb{Z}}(1+a_{k,n})\Bigr|\le(\sqrt{6}r+4K\pi^2r^2)/3,$$

where the constant K only depends on h.

Similarly, we find that

$$\Bigl|\frac{C(\sqrt{\lambda_n})}{\sqrt{\lambda_n}}\Bigr|=\Bigl|\prod_{k=1}^{\infty}\frac{\mu_k-\lambda_n}{\pi^2(k-\frac12)^2}\Bigr|=\prod_{k\in\mathbb{Z}}\frac{\omega_{2k-1}-\omega_{2n}}{\pi(k-\frac12)-\pi n}$$

and then mimic the above reasoning to establish the other uniform bound. The lemma is proved. □

As explained at the beginning of this Section, the above statements complete the proofs of Theorems 5.1 and 2.1.

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Appendix A. Riesz bases of sines and cosines

We recall (see, e.g., [15, Ch. 6] and [59, Ch. 4]) that a sequence $(f_n)_{n\in\mathbb{N}}$ in a separable Hilbert space H is called a Riesz basis of H if it is a homeomorphic image of an orthonormal basis of H. Then there are $M>0$ (the upper bound) and $m>0$ (the lower bound) such that, for every $f\in H$, we have

$$m\|f\|^2\le\sum\bigl|(f,f_n)\bigr|^2\le M\|f\|^2.$$

Riesz bases of $L_2(0,1)$ that are composed of exponential functions, or sines, or cosines, have been extensively studied in the literature starting from the early 1930-ies, see the books by Paley and Wiener [45] and Avdonin and Ivanov [5] for particulars and historical comments. For instance, the famous Kadets $\frac14$-theorem [31] implies that for

every $L < \frac{1}{4}$ there exist positive constants m and M such that as long as a sequence $(\omega_n)_{n\in\mathbb{Z}}$ of real numbers satisfies the condition

$$\sup_{n\in\mathbb{Z}} |\omega_n - \pi n| < \pi L, \tag{A.1}$$

then the sequence $(\mathrm{e}^{i\omega_n x})_{n\in\mathbb{Z}}$ of exponentials forms a Riesz basis of $L_2(-1,1)$ of upper bound M and lower bound m. Analogous results for families of sines and cosines were established in [18].

In this paper, we need generalizations of these results to sequences that may not satisfy condition (A.1). Recall that $\mathscr{L}^0(h,r)$, with $h \in (0,\pi)$ and $r > 0$, stands for the set of all strictly increasing sequences $\boldsymbol{\lambda} = (\omega_n^2)_{n\in\mathbb{N}}$ of positive numbers satisfying the conditions $\omega_1 > h$, $\omega_{n+1} - \omega_n \ge h$, $n \in \mathbb{N}$, and $\sum |\omega_n - \pi n|^2 \le r^2$. For every $\boldsymbol{\lambda} \in \mathscr{L}^0(h,r)$, we denote by $\mathscr{S}_{\boldsymbol{\lambda}}$ and $\mathscr{C}_{\boldsymbol{\lambda}}$ the sequences of functions $(\sin \omega_n x)_{n\in\mathbb{N}}$ and $(\cos \omega_n x)_{n\in\mathbb{Z}_+}$ respectively, with $\omega_0 := 0$. We also set $\omega_{-n} := -\omega_n$ and denote by $\mathscr{E}_{\boldsymbol{\lambda}}$ the sequence of functions $(\mathrm{e}^{\mathrm{i}\omega_n(1-2x)})_{n\in\mathbb{Z}}$. The following statement can be derived from the results of [24]:

Theorem A.1. *For every $h \in (0,\pi)$ and $r > 0$ there exist positive numbers M and m such that for every $\boldsymbol{\lambda} \in \mathscr{L}^0(h,r)$ the sequences $\mathscr{S}_{\boldsymbol{\lambda}}$, $\mathscr{C}_{\boldsymbol{\lambda}}$, and $\mathscr{E}_{\boldsymbol{\lambda}}$ are Riesz bases of $L_2(0,1)$ of upper bound M and lower bound m.*

Remark A.2. For $h \in (0,\pi)$ and $r > 0$, we denote by $\mathscr{M}^0(h,r)$ the set of increasing sequences $\boldsymbol{\mu} := (\mu_n)_{n=1}^\infty$ with the following properties:

(M1) $\mu_1 \ge 1$ and, for all $n \in \mathbb{N}$, $\sqrt{\mu_{n+1}} - \sqrt{\mu_n} \ge h$;
(M2) the numbers $\rho_n := \sqrt{\mu_n} - \pi(n - \frac{1}{2})$ form a sequence in ℓ_2 of norm at most r.

Then an analogue of the above theorem holds true for the family of sequences $\mathscr{C}_{\boldsymbol{\mu}} = (\cos \sqrt{\mu_n} x)_{n\in\mathbb{N}}$, with $\boldsymbol{\mu}$ running through the set $\mathscr{M}^0(h,r)$; see [24].

Appendix B. Sobolev spaces $W_2^s(0,1)$ and some of their properties

We recall here some facts about the Sobolev spaces $W_2^s(0,1)$ and Fourier coefficients of functions from these spaces. For details, we refer the reader to [35, Ch. 1].

B.1. **The definition.** By definition, the space $W_2^0(0,1)$ coincides with $L_2(0,1)$ and the norm $\|\cdot\|_0$ in $W_2^0(0,1)$ is just the $L_2(0,1)$-norm. For a natural l, the Sobolev space $W_2^l(0,1)$ consists of all functions f in $L_2(0,1)$, whose distributional derivatives $f^{(k)}$ for $k = 1,\dots,l$ also fall into $L_2(0,1)$. Being endowed with the norm

$$\|f\|_l := \Bigl(\sum_{k=0}^{l} \|f^{(k)}\|_0^2\Bigr)^{1/2}, \tag{B.1}$$

the space $W_2^l(0,1)$ becomes a Hilbert space.

The intermediate spaces $W_2^s(0,1)$ for arbitrary positive s can be constructed by interpolation [35, Ch. 1.2.1]. We shall need such spaces only for $s \in [0,2]$ and thus interpolate between $W_2^2(0,1)$ and $W_2^0(0,1)$ to get them, i.e.,

$$W_2^{2t}(0,1) := [W_2^2(0,1), W_2^0(0,1)]_{1-t}, \quad t \in (0,1).$$

The induced norms $\|\cdot\|_s$ (for $s = 1$ the norm (B.1) is equivalent to that defined by interpolation) are nondecreasing with $s \in [0,2]$, i.e., if $s < r$ and $f \in W_2^r(0,1)$, then $\|f\|_s \le \|f\|_r$. Since by construction the spaces $W_2^s(0,1)$ form an interpolation scale, the general interpolation theorem [35, Theorem 1.5.1] implies the following interpolation property for operators in these spaces.

Proposition B.1. *Assume that an operator T acts boundedly in $W_2^s(0,1)$ and $W_2^r(0,1)$, $s<r$. Then T is a bounded operator in $W_2^{ts+(1-t)r}(0,1)$ for every $t\in[0,1]$; moreover, $\|T\|_{ts+(1-t)r}\le\|T\|_s^t\|T\|_r^{1-t}$.*

Proposition B.1 yields boundedness in every $W_2^s(0,1)$, $s\in[0,2]$, of the reflection operator R given by $Rf(x)=f(1-x)$ and the operator M of multiplication by $\mathrm{i}x$, $Mf(x):=\mathrm{i}xf(x)$.

B.2. **Fourier transform.** For an arbitrary $f\in L_2(0,1)$ we denote by $\hat f:\mathbb{Z}\to\mathbb{C}$ its discrete Fourier transform, viz.

$$\hat f(n):=\int_0^1 f(x)\mathrm{e}^{-2\pi inx}\,dx.$$

The Fourier transform is a unitary mapping between $L_2(0,1)$ and $\ell_2(\mathbb{Z})$. We set

$$\tilde\ell_2^s(\mathbb{Z}):=\begin{cases}\ell_2^s(\mathbb{Z}) & \text{if} \quad s<\frac12,\\ \ell_2^s(\mathbb{Z})\dotplus\mathrm{ls}\{\mathbf{e}^{(1)}\} & \text{if} \quad \frac12\le s<\frac32,\\ \ell_2^s(\mathbb{Z})\dotplus\mathrm{ls}\{\mathbf{e}^{(1)},\mathbf{e}^{(2)}\}, & \text{if} \quad \frac32\le s\le 2,\end{cases}$$

where $\mathbf{e}^{(j)}$, $j=1,2$, denotes the sequence $(e_n^{(j)})_{n\in\mathbb{Z}}$ with $e_n^{(j)}:=n^{-j}$ for $n\ne0$ and $e_0^{(j)}:=0$. The norm $\|\cdot\|_s$ in $\hat\ell_2^s(\mathbb{Z})$ is defined as follows: given an element $\mathbf{x}:=(x_n)$ of $\ell_2^s(\mathbb{Z})$ and complex numbers a_1 and a_2 (with $a_1=a_2=0$ if $s<\frac12$ and $a_2=0$ if $\frac12\le s<\frac32$), we set

$$\|\mathbf{x}+a_1\mathbf{e}^{(1)}+a_2\mathbf{e}^{(2)}\|_s^2:=\sum_{n\in\mathbb{Z}}(1+n^2)^s|x_n|^2+|a_1|^2+|a_2|^2.$$

It is straightforward to verify that the Fourier transform of every function in $W_2^2(0,1)$ forms an element of $\tilde\ell_2^2(\mathbb{Z})$ and that $\|\hat f\|_2$ introduces a norm on $W_2^2(0,1)$ that is equivalent to that of (B.1). Since the spaces $\tilde\ell_2^s(\mathbb{Z})$, $s\in[0,2]$, form an interpolation scale, the interpolation theorem implies that, for every $s\in[0,2]$, the Fourier transform is a homeomorphism between the spaces $W_2^s(0,1)$ and $\tilde\ell_2^s(\mathbb{Z})$.

B.3. **Convolution.** As usual, $*$ denotes the convolution operation on $(0,1)$, viz.

$$(f*g)(x):=\int_0^1 f(x-t)g(t)\,dt;$$

here we extend a function f onto the interval $(-1,0)$ by periodicity, i.e., by setting $f(x):=f(x+1)$ for $x\in(-1,0)$. It is well known that convolution accumulates smoothness; the precise meaning of this statement is as follows.

Proposition B.2. *Assume that $s,t\in[0,1]$ and that $f\in W_2^s(0,1)$ and $g\in W_2^t(0,1)$ are arbitrary. Then the function $h:=f*g$ belongs to $W_2^{s+t}(0,1)$ and, moreover, there exists $C>0$ independent of f and g such that $\|h\|_{s+t}\le C\|f\|_s\|g\|_t$.*

Proof of this proposition is based on interpolation between the extreme cases $s,t=0,1$, which are handled with directly using the representation

$$(f*g)(x)=\int_0^x f(x-t)g(t)\,dt+\int_x^1 f(x-t+1)g(t)\,dt.$$

We also recall the relation $\widehat{f*g}(n)=\hat f(n)\hat g(n)$.

Appendix C. Some auxiliary results

Lemma C.1. *For f and g in $L_2(0,1)$, set*

$$\Phi(f,g) := \mathrm{V.p.} \sum_{n\in\mathbb{Z}} \hat{g}(n)[\exp\{\hat{f}(n)\mathrm{i}x\} - 1]\mathrm{e}^{2\pi \mathrm{i}nx};$$

then for every $s,t \in [0,1]$ the mapping

$$\Phi \,:\, W_2^s(0,1) \times W_2^t(0,1) \to W_2^{s+t}(0,1)$$

is analytic and Lipschitz continuous on bounded subsets.

Proof. If f and g are in $L_2(0,1)$, then the series for Φ converges absolutely, and thus $\Phi(f,g)$ is a continuous function. Take now $f \in W_2^s(0,1)$ and $g \in W_2^t(0,1)$. Developing $\exp\{\hat{f}(n)\mathrm{i}x\}$ into the Taylor series and changing the summation order (which is allowed since the resulting double series converges absolutely), we get

$$\Phi(f,g) = \sum_{k=1}^{\infty} \frac{(\mathrm{i}x)^k}{k!} \mathrm{V.p.} \sum_{n\in\mathbb{Z}} \hat{g}(n)\hat{f}^k(n)\mathrm{e}^{2\pi \mathrm{i}nx} = \sum_{k=1}^{\infty} \frac{(\mathrm{i}x)^k}{k!}\, h_k(x),$$

where $h_k := g * f^{<k>}$, $f^{<1>} := f$, and $f^{<k>}$ for $k \ge 2$ is the k-fold convolution of f with itself. By Proposition B.2, the functions h_k belong to $W_2^{s+t}(0,1)$, and

$$\|h_k\|_{s+t} \le C^{k+1}\|f\|_s^k \|g\|_t$$

with the constant C of that proposition. Also, the operator M of multiplication by $\mathrm{i}x$ is continuous in every $W_2^r(0,1)$, $r \in [0,2]$; denoting by $\|M\|_r$ the norm of the operator M in $W_2^r(0,1)$, we conclude that $\Phi(f,g)$ belongs to $W_2^{s+t}(0,1)$ and

$$\|\Phi(f,g)\|_{s+t} \le C\|g\|_t\bigl(\exp\{C\|M\|_{s+t}\|f\|_s\} - 1\bigr).$$

Moreover, every summand $(\mathrm{i}x)^k h_k/k!$ depends analytically on f and g; since the series converges absolutely in $W_2^{s+t}(0,1)$, analyticity of Φ follows.

By similar arguments we also find that

$$\Phi(f_1,g) - \Phi(f_2,g) = \sum_{k=1}^{\infty} \frac{(\mathrm{i}x)^k}{k!}\, g * (f_1^{<k>} - f_2^{<k>});$$

since

$$\|f_1^{<k>} - f_2^{<k>}\|_s \le kC^k\|f_1 - f_2\|_s\bigl(\|f_1\|_s + \|f_2\|_s\bigr)^{k-1},$$

it follows that

$$\|\Phi(f_1,g) - \Phi(f_2,g)\|_{s+t} \le C^2\|M\|_{s+t}\|f_1 - f_2\|_s\|g\|_t \exp\{C\|M\|_{s+t}(\|f_1\|_s + \|f_2\|_s)\}.$$

We observe now that the mapping Φ is linear in the second argument; therefore,

$$\begin{aligned}\|\Phi(f_1,g_1) - \Phi(f_2,g_2)\|_{s+t} &\le \|\Phi(f_1,g_1) - \Phi(f_2,g_1)\|_{s+t} + \|\Phi(f_2,g_1 - g_2)\|_{s+t}\\ &\le C_1\{\|f_1 - f_2\|_s + \|g_1 - g_2\|_t\},\end{aligned}$$

where $C_1 \le C^2\|M\|_{s+t}(1+K)\exp\{C\|M\|_{s+t}K\}$ as long as

$$\|f_1\|_s + \|f_2\|_s + \|g_1\|_t + \|g_2\|_t \le K,$$

and the desired Lipschitz continuity of Φ follows. ☐

We observe that

$$\Phi(f,g) + g = \text{V.p.} \sum_{n\in\mathbb{Z}} \hat{g}(n) \exp\{\hat{f}(n)\mathrm{i}x\} \mathrm{e}^{2\pi \mathrm{i} nx}$$

and hence the mapping

$$(f,g) \mapsto \text{V.p.} \sum_{n\in\mathbb{Z}} \hat{g}(n) \exp\{\hat{f}(n)\mathrm{i}x\} \mathrm{e}^{2\pi \mathrm{i} nx}$$

is uniformly continuous from $W_2^s(0,1) \times W_2^t(0,1)$ into $W_2^t(0,1)$.

Also, in the definition of Φ, we can formally take g to be the unity δ of the convolution algebra $L_2(0,1)$, which results in $\hat{g}(n) \equiv 1$. Slightly adapting the above proof, we get the following result.

Lemma C.2. *Fix $s \in [0,1]$ and, for $f \in W_2^s(0,1)$, set*

$$h(f) := \text{V.p.} \sum_{n\in\mathbb{Z}} [\exp\{\hat{f}(n)\mathrm{i}x\} - 1] \mathrm{e}^{2\pi \mathrm{i} nx}.$$

Then the function $h(f)$ belongs to $W_2^s(0,1)$ and the mapping $f \mapsto h(f)$ is analytic in $W_2^s(0,1)$ and Lipschitz continuous on bounded subsets.

We say that a function f is *odd* (resp. *even*) over $(0,1)$ (with respect to $\frac12$) if it satisfies the relation $f(1-x) = -f(x)$ (resp., the relation $f(1-x) = f(x)$) a.e. on $(0,1)$. For every integrable function f we define its odd part f_{odd} and even part f_{even} by the equalities

$$f_{\text{odd}}(x) := \frac{f(x) - f(1-x)}{2}, \qquad f_{\text{even}}(x) := \frac{f(x) + f(1-x)}{2}.$$

Since the reflection operator is continuous in every space $W_2^s(0,1)$, $s \ge 0$, the mappings $f \mapsto f_{\text{odd}}$ and $f \mapsto f_{\text{even}}$ are bounded in every $W_2^s(0,1)$. We denote by $W_{2,\text{odd}}^s(0,1)$ and $W_{2,\text{even}}^s(0,1)$ the subspaces of $W_2^s(0,1)$ consisting of functions that are respectively odd and even over $(0,1)$.

Corollary C.3. *Fix s and t in $[0,1]$. Then the mappings*

$$(f,g) \mapsto \sum_{n\in\mathbb{N}} c_{2n}(g) \cos[2\pi nx + 2s_{2n}(f)x]$$

from $W_{2,\text{odd}}^s(0,1) \times W_{2,\text{even}}^t(0,1)$ into $W_2^t(0,1)$ and

$$f \mapsto \sum_{n\in\mathbb{N}} \{\cos[2\pi nx + 2s_{2n}(f)x] - \cos 2\pi nx\}$$

from $W_{2,\text{odd}}^s(0,1)$ into $W_2^s(0,1)$ are analytic and Lipschitz continuous on bounded subsets.

Indeed, for an odd f we have $\hat{f}(n) = -\hat{f}(-n)$ and, as a result,

$$s_{2n}(f) = \frac{1}{2\mathrm{i}}\big[\hat{f}(-n) - \hat{f}(n)\big] = \mathrm{i}\hat{f}(n);$$

for an even g, we similarly have $c_{2n}(g) = \hat{g}(n) = \hat{g}(-n)$. Therefore the two series above coincide with $\frac12\Phi(2\mathrm{i}f,g) + \frac12 g$ and the function $\frac12 h(2\mathrm{i}f)$ of Lemma C.2 respectively, and the result follows.

Lemma C.4. *For f and g in $L_2(0,1)$, set*

$$\Psi(f,g) := \mathrm{V.p.}\sum_{n\in\mathbb{Z}}(-1)^n \int_0^1 g(t)\exp\{[\pi n+\hat f(n)]\mathrm{i}(1-2t)\}\,dt\,\mathrm{e}^{2\pi \mathrm{i} nx};$$

then for every $s\in[0,1]$ the mapping

$$\Psi\ :\ L_2(0,1)\times W_2^s(0,1)\to W_2^s(0,1)$$

is analytic and Lipschitz continuous on bounded subsets.

Proof. The coefficient of $\mathrm{e}^{2\pi \mathrm{i} nx}$ in the above series for Ψ can be written as

$$\int_0^1 g(t)\exp\{\hat f(n)\mathrm{i}(1-2t)\}\mathrm{e}^{-2\pi \mathrm{i} nt}\,dt \tag{C.1}$$

and gives the n-th Fourier coefficient of the function $h := \sum_{k=0}^\infty h_k/k!$, with $h_0 := g$, $h_k := f^{<k>} * M_1^k g$ for $k\ge 1$, and M_1 being the operator of multiplication by the function $\mathrm{i}(1-2t)$, i.e., $\Psi(f,g)=h$. Since h_k belongs to $W_2^s(0,1)$ and its norm there obeys the estimate

$$\|h_k\|_s \le C^k(\|f\|_0)^k\|M_1\|_s^k\|g\|_s$$

with C being the constant of Proposition B.2 and $\|M_1\|_s$ denoting the norm of the operator M_1 in the space $W_2^s(0,1)$, we conclude that the mapping Ψ is analytic. Its Lipschitz continuity on bounded subsets is established in the usual manner. □

As above, by taking an odd f and an even g of zero mean, we see that the expressions of (C.1) for n and $-n$ coincide with

$$2(-1)^n\int_0^1 g(t)\cos\{[\pi n+s_{2n}(-\mathrm{i}f)](1-2t)\}\,dt \tag{C.2}$$

and give the $2n$-th cosine Fourier coefficient of the function $h =: \Psi(f,g)\in W^s_{2,\mathrm{even}}(0,1)$.

References

[1] S. Albeverio, R. Hryniv, and Ya. Mykytyuk, Inverse spectral problems for Sturm–Liouville operators in impedance form, *J. Funct. Anal.* **222** (2005), 143–177.

[2] A. A. Alekseev, Stability of the inverse Sturm–Liouville problem for a finite interval, *Dokl. Akad. Nauk SSSR* **287** (1986), no. 1, 11–13 (in Russian).

[3] B. A. Ambartsumyan, Über eine Frage der Eigenwerttheorie, *Zeitschr. für Physik* **53** (1929), 690–695.

[4] L. Andersson, Inverse eigenvalue problems for a Sturm–Liouville equation in impedance form, *Inverse Probl.* **4** (1988), 929–971.

[5] S. A. Avdonin and S. A. Ivanov, *Families of Exponentials: the Method of Moments in Controllability Problems for Distributed Parameter Systems*, Cambridge Univ. Press, Cambridge, 1995.

[6] J. Bergh and J. Löfström, *Interpolation Spaces. An Introduction*, Grundlehren Math. Wiss., vol. **223**, Springer-Verlag, Berlin, 1976.

[7] G. Borg, Eine Umkehrung der Sturm–Liouvilleschen Eigenwertaufgabe, *Acta Math.* **78** (1946), no. 1, 1–96.

[8] C. F. Coleman and J. R. McLaughlin, Solution of the inverse spectral problem for an impedance with integrable derivative, I, *Comm. Pure Appl. Math.* **46** (1993), 145–184; II, *Comm. Pure Appl. Math.* **46** (1993), 185–212.

[9] S. Dineen, *Complex analysis on infinite-dimensional spaces*, *Springer Monographs in Mathematics*, Springer-Verlag London, Ltd., London, 1999.

[10] H. J. S. Dorren, E. J. Muyzert, and R. K. Snieder, The stability of one-dimensional inverse scattering, *Inverse Problems* **10** (1994), no. 4, 865–880.

[11] L. D. Faddeev, The inverse problem in the quantum theory of scattering, *Uspekhi Mat. Nauk* **14** (1959), pp. 57–119 (in Russian); *Engl. transl.: J. Math. Phys.* **4** (1963), 72–104.

[12] G. Freiling and V. Yurko, *Inverse Sturm-Liouville Problems and Their Applications*, Nova Sci. Publ. Corporation, Huntington, NY, 2001.

[13] M. G. Gasymov and B. M. Levitan, Determination of a differential operator from two spectra, *Uspekhi Matem. Nauk* **19** (1964), no. 2, 3–63 (in Russian).

[14] I. M. Gelfand and B. M. Levitan, On determination of a differential equation by its spectral function, *Izv. Akad. Nauk SSSR, Ser. Mat.* **15** (1951), no. 4, 309–360 (in Russian).

[15] I. Gohberg and M. Krein, *Introduction to the Theory of Linear Non-selfadjoint Operators in Hilbert Space*, Nauka Publ., Moscow, 1965 (in Russian); *Engl. transl.:* Amer. Math. Soc. Transl. Math. Monographs, vol. 18, Amer. Math. Soc., Providence, RI, 1969.

[16] I. Gohberg and M. Krein, *Theory of Volterra Operators in Hilbert Space and its Applications*, Nauka Publ., Moscow, 1967 (in Russian); *Engl. transl.:* Amer. Math. Soc. Transl. Math. Monographs, vol. 24, Amer. Math. Soc., Providence, RI, 1970.

[17] O. H. Hald, The inverse Sturm–Livouville problem with symmetric potentials, *Acta Math.* **141** (1978), no. 3–4, 263–291.

[18] X. He and H. Volkmer, Riesz bases of solutions of Sturm–Liouville equations, *J. Fourier Anal. Appl.*, **7** (2001), no. 3, 297–307.

[19] M. Hitrik, Stability of an inverse problem in potential scattering on the real line, *Commun. PDE* **25** (2000), 925–955.

[20] H. Hochstadt, The inverse Sturm–Liouville problem, *Comm. Pure Appl. Math.* **26** (1973), 715–729.

[21] H. Hochstadt, On the determination of the density of a vibrating string from spectral data, *J. Math. Anal. Appl.* **55** (1976), no. 3, 673–685.

[22] H. Hochstadt, On the well-posedness of the inverse Sturm–Liouville problems, *J. Differential Equations*, **23** (1977), no. 3, 402–413.

[23] M. Horvath and M. Kiss, Stability of direct and inverse eigenvalue problems for Schrödinger operators on finite intervals, *Int. Math. Res. Not.*, **2010** (2010), no. 11, 2022–2063.

[24] R. O. Hryniv, Uniformly bounded families of Riesz bases of exponentials, sines, and cosines, *Matem. Zametki* **87** (2010), no. 4, 542–553 (in Russian); *Engl. transl: Math. Notes* **87** (2010), no. 4, 510–520.

[25] R. O. Hryniv and Ya. V. Mykytyuk, Inverse spectral problems for Sturm–Liouville operators with singular potentials, *Inverse Problems* **19** (2003), 665–684.

[26] R. O. Hryniv and Ya. V. Mykytyuk, Inverse spectral problems for Sturm–Liouville operators with singular potentials, II. Reconstruction by two spectra, in *Functional Analysis and its Applications*, V. Kadets and W. Żelazko, eds., North-Holland Mathematics Studies, **197**, 97–114, North-Holland Publishing Co., Amsterdam, 2004.

[27] R. O. Hryniv and Ya. V. Mykytyuk, Transformation operators for Sturm–Liouville operators with singular potentials, *Math. Phys. Anal. Geom.* **7** (2004), 119–149.

[28] R. O. Hryniv and Ya. V. Mykytyuk, Inverse spectral problems for Sturm–Liouville operators with singular potentials, IV. Potentials in the Sobolev space scale, *Proc. Edinb. Math. Soc.* **49** (2006), no. 2, 309–329.

[29] R. O. Hryniv and Ya. V. Mykytyuk, Eigenvalue asymptotics for Sturm–Liouville operators with singular potentials, *J. Funct. Anal.* **238** (2006), no. 1, 27–57.

[30] R. O. Hryniv and Ya. V. Mykytyuk, On zeros of some entire functions, *Trans. Amer. Math. Soc.* **361** (2009), no. 4, 2207–2223.

[31] M. I. Kadets, The exact value of the Paley–Wiener constant, *Dokl. Akad. Nauk SSSR* **155** (1964) no. 6, 1253–1254 (in Russian); *Engl. transl.: Soviet Math. Dokl.* **5** (1964), 559–561.

[32] E. Korotyaev, Stability for inverse resonance problem, *Int. Math. Res. Not.* **73** (2004), 3927–3936.

[33] M. G. Krein, On the inverse problem for a non-homogeneous cord, *Dokl. Akad. Nauk SSSR* **82** (1951), 669–672 (in Russian).

[34] N. Levinson, The inverse Sturm–Liouville problem, *Mat. Tidsskr. B* **25** (1949), 25–30.

[35] J.-L. Lions and E. Magenes, *Non-Homogeneous Boundary Value Problems and Applications,* I, Springer Verlag, Berlin—Heildelberg—New York, 1972.

[36] M. M. Malamud, *Spectral Analysis of Volterra Operators and Inverse Problems for Systems of Ordinary Differential Equations*, SFB 288 Preprint No. 269, Berlin, June 1997.

[37] V. A. Marchenko, Some questions of the theory of second order differential operators, *USSR Doklady* **72** (1950), no. 3, 457–460 (in Russian).

[38] V. A. Marchenko, *Sturm–Liouville Operators and Their Applications*, Naukova Dumka Publ., Kiev, 1977 (in Russian); *Engl. transl.:* Birkhäuser Verlag, Basel, 1986.

[39] V. A. Marchenko and K. V. Maslov, Stability of the problem of the reconstruction of the Sturm–Liouville operator in terms of the spectral function, *Mat. Sbornik (N.S.)* **81(123)** (1970), 525–551 (in Russian).

[40] M. Marletta and R. Weikard, Weak stability for an inverse Sturm–Liouville problem with finite spectral data and complex potential, *Inverse Problems* **21** (2005), no. 4, 1275–1290.

[41] M. Marletta and R. Weikard, Stability for the inverse resonance problem for a Jacobi operator with complex potential, *Inverse Problems* **23** (2007), no. 4, 1677–1688.

[42] M. Marletta, R. Shterenberg, and R. Weikard, On the inverse resonance problem for Schrödinger operators, *Comm. Math. Phys.* **295** (2010), no. 2, 465–484.

[43] J. R. McLaughlin, Stability theorems for two inverse problems, *Inverse Probl.* **4** (1988), 529–540.

[44] A. Mizutani, On the inverse Sturm–Liouville problem, *J. Fac. Sci. Univ. Tokyo Sect. IA Math.* **31** (1984), no. 2, 319–350.

[45] R. Paley and N. Wiener, *Fourier Transforms in the Complex Domain*, American Mathematical Society, Colloquium Publications, Vol. XIX, Providence, R.I., 1934.

[46] J. Pöschel and E. Trubowitz, *Inverse Spectral Theory*, Academic Press, Orlando, Florida, 1987 (Pure and Applied Math., Vol. 130).

[47] W. Rundell and P. E. Sacks, The reconstruction of Sturm–Liouville operators, *Inverse Problems* **8** (1992), 457–482.

[48] T. I. Ryabushko, Stability of the reconstruction of a Sturm–Liouville operator from two spectra, II, *Teor. Funktsiĭ Funktsional. Anal. i Prilozhen.* **18** (1973), 176–185 (in Russian).

[49] T. I. Ryabushko, Stability of the reconstruction of a Sturm–Liouville boundary value problem on a semiaxis from two spectra, *Teor. Funkts. Funktsion. Anal. Prilozhen.* **35** (1981), 96–100 (in Russian).

[50] T. I. Ryabushko, Estimation of the norm of the difference of two potentials of Sturm–Liouville boundary value problems, *Teor. Funkts. Funktsion. Anal. Prilozhen.* **39** (1983), 114–117 (in Russian).

[51] A. M. Savchuk, On eigenvalues and eigenfunctions of Sturm–Liouville operators with singular potentials, *Matem. Zametki* **69** (2001), no. 2, 277–285 (in Russian); *Engl. transl.: Math. Notes* **69** (2001), no. 1–2, 245–252.

[52] A. M. Savchuk and A. A. Shkalikov, Sturm–Liouville operators with singular potentials, *Matem. Zametki* **66** (1999), no. 6, 897–912 (in Russian); *Engl. transl.: Math. Notes* **66** (1999), no. 5–6, 741–753.

[53] A. M. Savchuk and A. A. Shkalikov, Sturm–Liouville operators with distributional potentials, *Trudy Mosk. Matem Ob-va*, **64** (2003), 159–212 (in Russian); *Engl. transl.: Trans. Moscow Math. Soc.* **2003**, 143–192.

[54] A. M. Savchuk and A. A. Shkalikov, Inverse problem for Sturm–Liouville operators with distribution potentials: reconstruction from two spectra, *Russ. J. Math. Phys.* **12** (2005), no. 4, 507–514.

[55] A. M. Savchuk and A. A. Shkalikov, On the eigenvalues of the Sturm–Liouville operator with potentials in Sobolev spaces, *Matem. Zametki* **80** (2006), no. 6, 864–884 (in Russian); *Engl. transl.: Math. Notes* **80** (2006), no. 5–6, 814–832.

[56] A. M. Savchuk and A. A. Shkalikov, On the properties of mappings associated with inverse Sturm–Liouville problems, *Tr. Mat. Inst. Steklova* **260** (2008), 227–247 (in Russian).

[57] A. M. Savchuk and A. A. Shkalikov, Inverse problems for Strum–Liouville operators with potentials in Sobolev spaces: Uniform stability, *Funktsion. Anal. Prilozhen.* **44** (2010), no. 4, 34–53 (in Russian); *Engl. transl.: Funct. Anal. Appl.* **44** (2010), no. 4, 270–285.

[58] L. Tartar, Interpolation non linéaire et régularité, *J. Functional Analysis* **9** (1972), 469–489.

[59] R. Young, *An Introduction to Nonharmonic Fourier Series*, Academic Press, New York, 2001.

[60] V. A. Yurko, On stability of recovering the Sturm–Liouville operators, *Diff. Equat. Theory Functions (Saratov Univ.)* **3** (1980), 113–124 (in Russian).

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